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# THE ELEMENTS OF EUCLID

FOR THE USE OF SCHOOLS AND COLLEGES

*WITH NOTES, AN APPENDIX, AND EXERCISES BY*

I. TODHUNTER, D.Sc., F.R.S.

NEW EDITION, REVISED AND ENLARGED, BY

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## PREFACE.

IN preparing a new edition of the late Dr. Todhunter's *Euclid*, the following are the principal alterations and additions that I have made :

(1) The text of the Propositions has been simplified and shortened. A sparing use has been made of symbols in the place of constantly recurring words. Some of the proofs (*e.g.* II. 13) have been altered in accordance with modern usage, but this has always been done subject to the regulations at present in force in the Universities of Oxford and Cambridge.

(2) Considerable trouble has been taken to arrange the book so that, with very few exceptions, each Proposition is commenced on a fresh page, and may be read by the student without his turning over a leaf.

(3) The proofs of Book V. have been much shortened. To them, and also to the proofs of Book II., the corresponding algebraic formulae have been added.

(4) The more important of Dr. Todhunter's Notes have been appended to the Propositions to which they refer.

(5) The total number of Exercises has been doubled.

(6) The easier of the Exercises in the previous edition, and a large number of additional ones, have been classified and follow, in the Text, the Propositions on which they depend. The more important of these have asterisks prefixed to them, and, as far as possible, their results should

be remembered by the student as part of his geometrical knowledge.

(7) All through the book, with the exception of the Exercises at the end, which are left to the student, hints have been annexed to the more difficult and more important Exercises.

(8) The Appendix has been more than doubled in quantity, and the theorems in it have been classified according to the Book to which they refer and on which they depend; a large number of Exercises has been incorporated in it.

(9) Sections have been added dealing with Poles and Polars, Orthogonal Circles, Pedal Triangles, The Pedal Line, The Nine-Point Circle, Co-axal Circles, Harmonic Ranges, Inversion, and the Properties of a Complete Quadrilateral.

Out of the large number of Propositions on these subjects it is clear that only a selection could be made, but I have endeavoured within the range chosen to omit no important Proposition. Without unduly increasing the size of the book it was impossible to prove all such important Propositions in the text. It is hoped, however, that the hints attached to those given as Exercises will be sufficient for any fairly intelligent student.

For any corrections of misprints or errors, or any suggestions for improvement, I shall be very grateful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE, EGHAM,  
SURREY, *April 21st*, 1899.



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# EUCLID'S ELEMENTS.

## BOOK I.

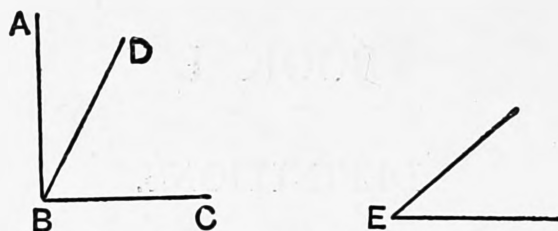
### DEFINITIONS.

1. A **point** is that which has position but no magnitude.
2. A **line** is that which has length without breadth.
3. The extremities of a line are points, and the intersection of two lines is a point.
4. A **straight line** is one which lies evenly between its extreme points.
5. A superficies, or **surface**, is that which has only length and breadth.
6. The boundaries of a surface are lines.
7. A plane surface, or a **plane**, is that in which any two points being taken, the straight line between them lies wholly in that surface.  
[Thus if we take a piece of wood, one of whose edges is straight, and apply it to the surface, and find that the edge fits closely to the surface everywhere, we know that the surface is a plane.]
8. A **plane angle** is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.
9. A **plane rectilineal angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

The point where the two straight lines meet is called the **vertex** of the angle, and the straight lines themselves are sometimes called the **arms**.



*Note.* When several angles are at one point B, any one of them is expressed by three letters, of which the letter which is at the vertex of the angle is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus, the angle which is contained

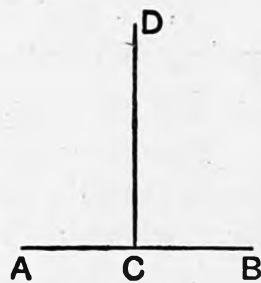


by the straight lines AB, CB is named the angle ABC, or CBA ; the angle which is contained by the straight lines AB, DB is named the angle ABD, or DBA ; and the angle which is contained by the straight lines DB, CB is named the angle DBC, or CBD ; but if there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle E.

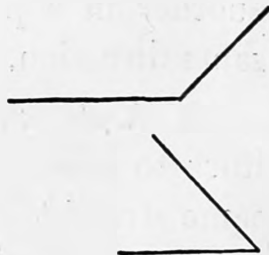
Two such angles as ABD, DBC, which are on opposite sides of one common bounding line BD, are called **adjacent** angles.

The beginner must carefully observe that no change is made in an angle by prolonging the lines that form it, that is, by altering the length of its arms.

**10.** When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle** ; and the straight line which stands on the other is called a **perpendicular** to it.



**11.** An **obtuse angle** is an angle which is greater than a right angle.

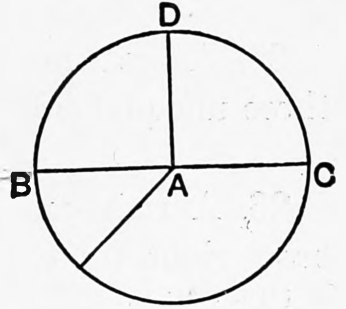


**12.** An **acute angle** is an angle which is less than a right angle.

**13.** A **plane figure** is one which is enclosed by one or more bounding lines, straight or curved ; and the sum of these bounding lines is called a **perimeter**.

**14.** If the boundaries consist of straight lines only, it is called a **plane rectilineal figure**, and these straight lines are called its **sides**.

**15.** A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another :



Also this point is called the **centre** of the circle.

**16.** A **radius** of a circle is a straight line drawn from the centre to the circumference.

**17.** A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

**18.** A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

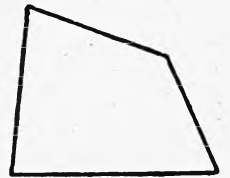
[Thus, in the above figure, A is the centre ; AB, AD, AC are radii ; BC is a diameter ; and the figure bounded by the straight line BC and the curved line BDC is a semicircle.]

**19.** A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

**20.** A **triangle** is a plane figure contained by three straight lines. [Any angular point may be called a **vertex** and the opposite side the **base**.]

**21.** A **quadrilateral** is a plane figure contained by four straight lines.

[The straight line joining two opposite corners of a quadrilateral is called a **diagonal**.]



**22.** A **polygon** is a plane figure contained by more than four straight lines.

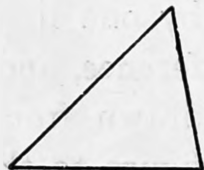
**23.** An **equilateral triangle** is one which has three equal sides.



**24.** An **isosceles triangle** is one which has two sides equal.



**25.** A **scalene triangle** is one which has three unequal sides.

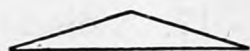


**26.** A **right-angled triangle** is one which has a right angle.

[The side opposite to the right angle in a right-angled triangle is frequently called the **hypotenuse**.]



**27.** An **obtuse-angled triangle** is one which has an obtuse angle.

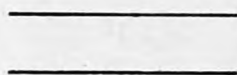


**28.** An **acute-angled triangle** is one which has three acute angles.



[It will be seen later that every triangle has at least two acute angles.]

**29.** **Parallel straight lines** are such as are in the same plane, and which being produced ever so far both ways do not meet.



**30.** A **parallelogram** is a four-sided figure which has its opposite sides parallel.

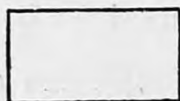


**31.** A **square** is a four-sided figure which has all its sides equal, and one of its angles a right angle.



**32.** A **rectangle** is a parallelogram which has one of its angles a right angle.

[It will be shewn (see Note, Prop. 46) that all the angles of a rectangle or a square are right angles.]





**33.** A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.



**34.** A **rhomboid** is a four-sided figure which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

**35.** A **trapezium** is a four-sided figure which has two of its sides parallel.

## POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point :

2. That a terminated straight line may be produced to any length in a straight line : and

3. That a circle may be described with any centre, at any distance from that centre, that is, with any given line drawn from the centre as radius.

*Note.* The postulates state what processes we assume that we can effect. It is sometimes stated that the postulates amount to requiring the use of a *ruler* and *compasses*. It must, however, be observed that the ruler is not supposed to be a *graduated* ruler, so that we cannot use it to measure off assigned lengths. Also, we are not supposed to use the compasses for the purpose of transferring any distance from one part of a figure to another ; in other words, the compasses may be supposed to close of themselves, as soon as one of their points is removed from the paper. [After Prop. 3 it will be found that this restriction is no longer necessary.]

## AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. The whole is greater than its part.

9. Magnitudes which can be made to coincide with one another are equal to one another.

[This method of placing one geometrical magnitude upon a second is called the method of **superposition**, and the first magnitude is said to be **applied** to the other.]

10. Two straight lines cannot enclose a space.

This axiom should be extended thus :

*If two straight lines coincide in two points, they must coincide both beyond and between these points.*

11. All right angles are equal to one another.

[This axiom admits of proof ; see Note to I. 14.]

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

*Note.* An axiom is a truth which can be taken for granted, and requires no proof. The axioms are called in the original *Common Notions*.

The first eight are true of magnitudes of all kinds, and are sometimes called General Axioms; the remainder refer exclusively to geometrical magnitudes, and are sometimes called Geometrical Axioms.

The fourth axiom is sometimes referred to in editions of Euclid when in reality more is required than this axiom expresses. Euclid says that if A and B be unequal, and C and D equal, the sum of A and C is *unequal* to the sum of B and D. What Euclid often requires is something more, namely, that if A be greater than B, and C and D be equal, the sum of A and C is *greater* than the sum of B and D. Such an axiom as this is required, for example, in I. 17. A similar remark applies to the fifth axiom.

The eleventh axiom is not, strictly speaking, an axiom, for it can be proved; it is not required before I. 14, and the twelfth axiom is not required before I. 29; we shall not consider these axioms until we arrive at the propositions in which they are respectively required for the first time.

### NOTE ON EUCLID'S PROPOSITIONS.

Euclid divides his different books into separate propositions, each proposition being derived from previous propositions. Propositions are of two kinds, **Problems** and **Theorems**.

In a problem Euclid states some definite construction which is to be made, such as to draw some particular figure. When this construction has been made the problem is *solved*.

In a theorem Euclid states some definite geometrical fact which he proceeds to prove. He names first the **Hypothesis**, or the conditions which he assumes, and then the **Conclusion** which he asserts will follow.

In a proposition we usually have :

(1) The *General Enunciation*, which is a preliminary statement.  
 (2) The *Particular Enunciation*, which applies the general enunciation to a particular diagram.

(3) The *Construction*, which shows what drawing of lines, etc., he wants.

(4) The *Demonstration*, or *Proof*, which shows that the problem has been solved, or that the theorem is true.

The letters Q.E.F. at the end of a problem stand for *Quod erat Faciendum*, that is, *which was to be done*.

The letters Q.E.D. at the end of a theorem stand for *Quod erat Demonstrandum*, that is, *which was to be proved*.

A **Corollary** is a statement which follows immediately from the proposition to which it is appended, and which thus requires no further proof.

Many of the corollaries are not in the original text but have been introduced by various editors.

## SYMBOLS AND ABBREVIATIONS.

The following symbols are often used for words and expressions that continually occur in Euclid's Proofs. They may be used by the student in writing out propositions, and some of the more common will be gradually introduced in this Edition.

$\therefore$  for Therefore.

$=$  for Equals or Is equal to or Are equal to.

$>$  for Is greater than.  $<$  for Is less than.

$\angle$  and  $\angle^s$  for Angle and Angles.

$\triangle$  and  $\triangle^s$  for Triangle and Triangles.

$\perp^r$  for Perpendicular to.

$\parallel^l$  for Parallel.  $\parallel^m$  and  $\text{Par}^m$  for Parallelogram.

Rt.  $\angle$  for Right Angle.  $\odot$  for Circle or Circumference.

The symbol  $\because$  is sometimes used for Because. It is, however, liable to be confounded with the symbol for Therefore, and will not be used in this Edition.

In addition any well understood abbreviations for words may be used; e.g.:

Def. for Definition; Ax. for Axiom; Post. for Postulate; Constr. for Construction; Hyp. for Hypothesis; Pt. for Point; Str. for Straight; Perp. for Perpendicular; Alt. for Altitude; Sq. for Square; Rect. for Rectangle; Quad<sup>l</sup> for Quadrilateral; Adj. for Adjacent.

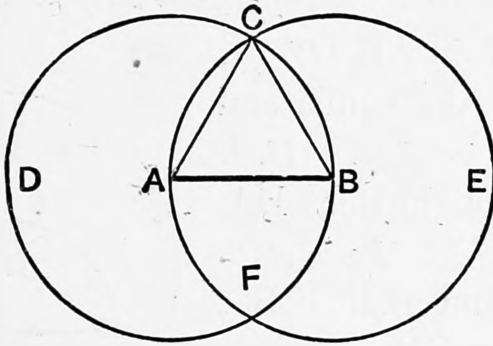


## PROPOSITION 1. PROBLEM.

*To describe an equilateral triangle on a given finite straight line.*

Let  $AB$  be the given straight line :

*it is required to describe an equilateral triangle on  $AB$ .*



**Construction.** With centre  $A$  and radius  $AB$ , describe the circle  $BCD$ . [Postulate 3.]

With centre  $B$  and radius  $BA$ , describe the circle  $ACE$ .

[Postulate 3.]

From the point  $C$ , at which the circles cut one another, draw the straight lines  $CA$  and  $CB$ . [Postulate 1.]

$ABC$  shall be an equilateral triangle.

**Proof.** Because  $A$  is the centre of the circle  $BCD$ ,

$AC$  is equal to  $AB$ .

[Definition 15.]

Also because  $B$  is the centre of the circle  $ACE$ ,

$BC$  is equal to  $BA$ .

[Definition 15.]

But it has been shewn that  $CA$  is equal to  $AB$ ;

therefore  $CA$  and  $CB$  are each of them equal to  $AB$ .

But things which are equal to the same thing are equal to one another ;

[Axiom 1.]

therefore  $CA$  is equal to  $CB$ .

Therefore  $CA$ ,  $AB$ ,  $BC$  are equal to one another.

Wherefore the triangle  $ABC$  is equilateral,

[Definition 23.]

and it is described on the given straight line  $AB$ .

[Q. E. F.]

## PROPOSITION 2. PROBLEM.

*From a given point to draw a straight line equal to a given straight line.*

Let  $A$  be the given point, and  $BC$  the given straight line: *it is required to draw from the point  $A$  a straight line equal to  $BC$ .*

**Construction.** Join  $AB$ , [*Post. 1.*]  
and on it describe the equilateral triangle  $DAB$ , [*I. 1.*

and produce the straight lines  $DA$ ,  $DB$  to  $E$  and  $F$ . [*Postulate 2.*

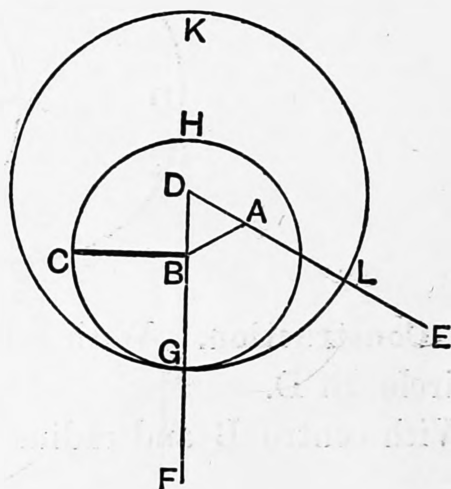
With centre  $B$  and radius  $BC$ , describe the circle  $CGH$ ,

meeting  $DF$  at  $G$ . [*Postulate 3.*

With centre  $D$  and radius  $DG$ , describe the circle  $GLK$ ,

meeting  $DE$  at  $L$ . [*Postulate 3.*

$AL$  shall be equal to  $BC$ .



**Proof.** Because the point  $B$  is the centre of the circle  $CGH$ ,  
 $BC$  is equal to  $BG$ . [*Definition 15.*

Also because  $D$  is the centre of the circle  $GLK$ ,

$DL$  is equal to  $DG$ ; [*Definition 15.*

and  $DA$ ,  $DB$  parts of them are equal; [*Definition 23.*

therefore the remainders  $AL$ ,  $BG$  are equal. [*Axiom 3.*

But it has been shewn that  $BC$  is equal to  $BG$ ;

therefore  $AL$  and  $BC$  are each of them equal to  $BG$ .

But things which are equal to the same thing are equal to one another. [*Axiom 1.*

Therefore  $AL$  is equal to  $BC$ .

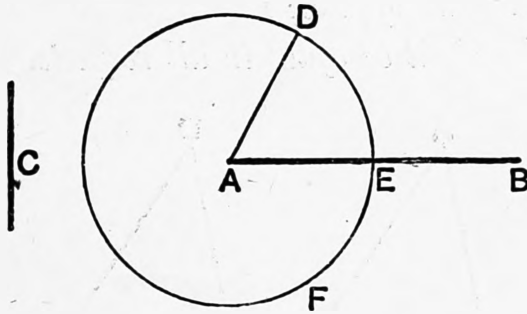
Wherefore from the given point  $A$  a straight line  $AL$  has been drawn equal to the given straight line  $BC$ . [*Q.E.F.*

## PROPOSITION 3. PROBLEM.

*From the greater of two given straight lines to cut off a part equal to the less.*

Let  $AB$  and  $C$  be the two given straight lines, of which  $AB$  is the greater :

*It is required to cut off from  $AB$ , the greater, a part equal to  $C$ , the less.*



**Construction.** From the point  $A$  draw the straight line  $AD$  equal to  $C$  ; [I. 2.  
and with centre  $A$  and radius  $AD$ , describe the circle  $DEF$ ,  
meeting  $AB$  at  $E$ . [Postulate 3.  
Then  $AE$  shall be equal to  $C$ .

**Proof.** Because  $A$  is the centre of the circle  $DEF$ ,  
therefore  $AE$  is equal to  $AD$ . [Def. 15.

But  $C$  is equal to  $AD$ . [Construction.

Therefore  $AE$  and  $C$  are each of them equal to  $AD$ .

Therefore  $AE$  is equal to  $C$ .

[Axiom 1.

Wherefore from  $AB$ , the greater of two given straight lines, a part  $AE$  has been cut off equal to  $C$ , the less.

## EXERCISES.

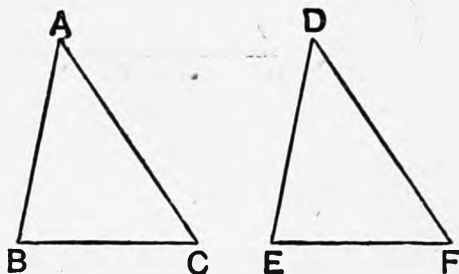
1. In the figure of I. 1, if the two circles meet again in  $F$ , prove that  $ACBF$  is a rhombus.

2. On a given straight line describe an isosceles triangle, having each of the sides equal to a given straight line.

3. On a given straight line describe an isosceles triangle, having each of the sides double the base.

## PROPOSITION 4. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another ;  
they shall also have their bases or third sides equal ;  
the two triangles shall be equal in area ,  
and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite ;  
that is, the triangles shall be equal in all respects.*



Let  $ABC$ ,  $DEF$  be two triangles which have  
the side  $AB$  equal to the side  $DE$ ,  
the side  $AC$  equal to the side  $DF$ ,  
and the angle  $BAC$  equal to the angle  $EDF$  ;  
*then shall the base  $BC$  be equal to the base  $EF$ ,  
the triangle  $ABC$  shall be equal in area to the triangle  $DEF$ ,  
and the other angles shall be equal, each to each, to which the equal sides are opposite, namely, the angle  $ABC$  to the angle  $DEF$ , and the angle  $ACB$  to the angle  $DFE$ .*

**Proof.** For if the triangle  $ABC$  be applied to the triangle  $DEF$ , so that the point  $A$  may be on the point  $D$ , and the straight line  $AB$  on the straight line  $DE$ ,

the point  $B$  will coincide with the point  $E$ ,

because  $AB$  is equal to  $DE$ .

[*Hypothesis.*

And because  $AB$  coincides with  $DE$ ,

and the angle  $BAC$  is equal to the angle  $EDF$ ,

[*Hypothesis.*

therefore  $AC$  will fall on  $DF$ .



Therefore also the point C will coincide with the point F,  
because AC is equal to DF. [*Hypothesis.*

But the point B was shewn to coincide with the point E,  
therefore the base BC will coincide with the base EF ;  
for if not, two straight lines will enclose a space, which is  
impossible. [*Axiom 10.*

Therefore the base BC coincides with the base EF, and is  
equal to it. [*Axiom 9.*

Therefore the whole triangle ABC coincides with the whole  
triangle DEF, and is equal to it. [*Axiom 9.*

And the other angles of the one coincide with the other  
angles of the other, and are equal to them, namely,

the angle ABC to the angle DEF,  
and the angle ACB to the angle DFE.

Wherefore, *if two triangles, etc.* [Q. E. D.

*Note.* Triangles, such as ABC and DEF, which are equal in all  
respects, are said to be **Congruent**.

### EXERCISES.

1. If two straight lines, AB and CD, bisect one another at right  
angles in O, any point P in either of them, AB, is equidistant from the  
ends, C and D, of the other.

\*\*2. If the straight line drawn from the vertex of a triangle to the  
middle point of the base cuts the base at right angles, the triangle is  
isosceles.

3. D and E are the middle points of the equal sides AB, AC of an  
isosceles triangle ABC ; prove that CD and BE are equal.

4. ABCD is a quadrilateral, and its diagonals bisect one another at  
right angles ; prove that ABCD is a rhombus.

5. The sides AB, AD of a quadrilateral ABCD are equal, and the  
diagonal AC bisects the angle BAD ; prove that the sides CB, CD are  
equal, and that the diagonal AC bisects the angle BCD.

6. ABCD and EFGH are quadrilaterals, such that the sides AB,  
BC, CD are equal respectively to the sides EF, FG, GH, and the  
angles ABC, BCD are equal respectively to the angles EFG, FGH.  
Prove that the quadrilaterals are equal in all respects.

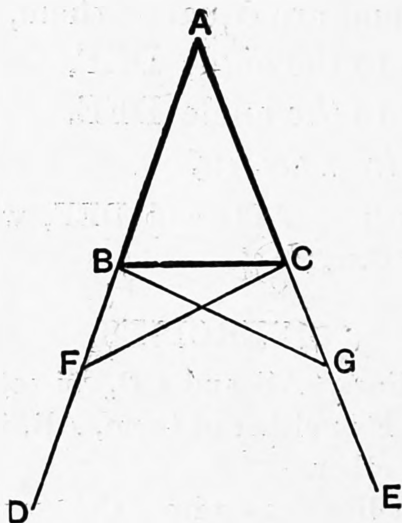
\*\*7. The straight line which bisects the vertical angle of an isosceles  
triangle bisects the base, and is also perpendicular to it.

## PROPOSITION 5. THEOREM.

*The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.*

Let  $ABC$  be an isosceles triangle, having the side  $AB$  equal to the side  $AC$ , and let the straight lines  $AB$ ,  $AC$  be produced to  $D$  and  $E$ :

*then the angle  $ABC$  shall be equal to the angle  $ACB$ , and the angle  $CBD$  to the angle  $BCE$ .*



• **Construction.** In  $BD$  take any point  $F$ , and from  $AE$  the greater cut off  $AG$  equal to  $AF$  the less. [I. 3. Join  $FC$  and  $GB$ .

**Proof.** (1) In the two triangles  $AFC$ ,  $AGB$ ,  
 because  $\begin{cases} AF \text{ is equal to } AG, & [\text{Construction.}] \\ \text{and } AC \text{ is equal to } AB, & [\text{Hypothesis.}] \\ \text{and the contained angle } FAG \text{ is common to both;} \end{cases}$   
 therefore the triangles are equal in all respects, so that  
 the base  $FC$  is equal to the base  $GB$ ,  
 the angle  $ACF$  is equal to the angle  $ABG$ ,  
 and the angle  $AFC$  to the angle  $AGB$ . [I. 4.]

(2) Because the whole  $AF$  is equal to the whole  $AG$ , of which the parts  $AB$ ,  $AC$  are equal, [*Hypothesis.*  
 the remainder  $BF$  is equal to the remainder  $CG$ . [*Axiom 3.*

(3) Then in the triangles  $BFC$ ,  $CGB$ ,  
 because  $\left\{ \begin{array}{l} BF \text{ is equal to } CG, \\ \text{and } FC \text{ is equal to } GB, \\ \text{and the angle } BFC \text{ is equal to the angle } CGB; \end{array} \right.$  [*Proved in (2).*  
[*Proved in (1).*  
[*Proved in (1).*

therefore the triangles are equal in all respects, [*I. 4.*  
 so that the angle  $FBC$  is equal to the angle  $GCB$ ,  
 and the angle  $BCF$  to the angle  $CBG$ .

(4) Since the whole angle  $ABG$  is equal to the whole angle  $ACF$ , [*Proved in (1).*  
 and that the parts of these, the angles  $CBG$ ,  $BCF$  are also equal; [*Proved in (3).*  
 therefore the remaining angle  $ABC$  is equal to the remaining angle  $ACB$ , [*Axiom 3.*  
 and these are the angles at the base of the triangle  $ABC$ .

Also it has been shewn that the angle  $FBC$  is equal to the angle  $GCB$ , and these are the angles on the other side of the base.

Wherefore, *the angles, etc.* [*Q.E.D.*

**Corollary.** Every equilateral triangle is also equiangular.

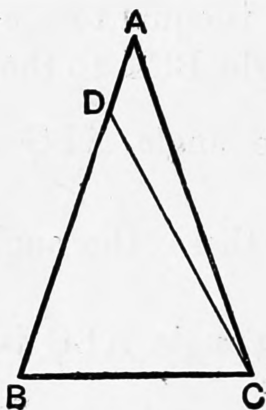
*Note.* This proposition is often found hard by beginners. This probably arises from the fact that the triangles  $ACF$ ,  $ABG$  overlap one another; as also do the triangles  $BCF$  and  $CBG$ . For the part (1) of the proof the student is recommended to draw separately the two triangles  $ACF$  and  $ABG$ , and consider the figures thus obtained; for the part (3) of the proof he should similarly draw separately the two triangles  $BCF$  and  $CBG$ .

## PROPOSITION 6. THEOREM.

*If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.*

Let  $ABC$  be a triangle, having the angle  $ABC$  equal to the angle  $ACB$  :

*then the side  $AC$  shall be equal to the side  $AB$ .*



**Construction.** For if  $AC$  be not equal to  $AB$ , one of them must be greater than the other.

Let  $AB$  be the greater, and from it cut off  $DB$  equal to  $AC$  the less,

[I. 3.]

and join  $DC$ .

**Proof.** In the triangles  $DBC$ ,  $ACB$ ,

because  $\begin{cases} DB \text{ is equal to } AC, \\ \text{and } BC \text{ is common to both triangles,} \\ \text{and the angle } DBC \text{ is equal to the angle } ACB, \end{cases}$

[Construction.]

[Hypothesis.]

therefore the triangles are equal in all respects,

[I. 4.]

the less to the greater ; which is absurd.

[Axiom 8.]

Therefore  $AB$  is not unequal to  $AC$ , that is, it is equal to it.

Wherefore, *if two angles, etc.*

[Q. E. D.]

**Corollary.** Every triangle, which has three equal angles, is also equilateral.

## NOTE TO PROPOSITION 6.

**Converse Theorem.** One proposition is said to be the converse of another when the conclusion of each is the hypothesis of the other. I. 6 is the *converse* of part of I. 5. Thus in I. 5 the hypothesis is the equality of the sides, and one conclusion is the equality of the angles; in I. 6 the hypothesis is the equality of the angles and the conclusion is the equality of the sides.

The converse of a true proposition is not necessarily true; the student however will see, as he proceeds, that Euclid shews that the converses of many geometrical propositions are true.

I. 6 is an example of the **indirect** mode of demonstration, in which a result is established by shewing that some absurdity follows from supposing the required result to be untrue. Hence this mode of demonstration is called the **reductio ad absurdum**. Indirect demonstrations are often less esteemed than direct demonstrations; they are said to shew that a theorem is true rather than to shew *why* it is true. Euclid uses the *reductio ad absurdum* chiefly when he is demonstrating the converse of some former theorem; see I. 14, 19, 25, 40.

## EXERCISES.

1. If the angles  $ABC$ ,  $ACB$  at the base of an isosceles triangle  $ABC$  are bisected by the straight lines  $BD$  and  $CD$ , which meet at  $D$ , prove that  $BDC$  will be an isosceles triangle.

2.  $BAC$  is a triangle having the angle  $B$  double of the angle  $A$ . If  $BD$  bisects the angle  $B$  and meets  $AC$  at  $D$ , prove that  $BD$  and  $AD$  are equal.

In the figure of I. 5, if  $FC$  and  $BG$  meet at  $H$ , prove that

3.  $BH$  and  $CH$  are equal.

4.  $AH$  bisects each of the angles  $BAC$ ,  $BHC$ .

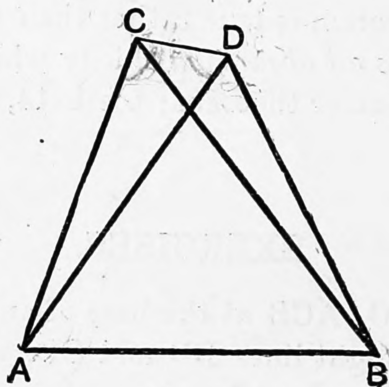
5.  $AH$  bisects  $BC$  at right angles.

6. If on the sides  $AB$ ,  $BC$ ,  $CA$  of an equilateral triangle  $ABC$  equal lengths  $AP$ ,  $BQ$ ,  $CR$  be measured, prove that  $PQR$  is also an equilateral triangle.

## PROPOSITION 7. THEOREM.

*On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.*

If it be possible, on the same base  $AB$ , and on the same side of it, let there be two triangles  $ACB$ ,  $ADB$ , having their sides  $CA$ ,  $DA$ , which are terminated at the extremity  $A$  of the base, equal to one another, and also their sides  $CB$ ,  $DB$ , which are terminated at  $B$ , equal to one another.



**Construction.** Join  $CD$ .

**Proof.** CASE I. Let the vertex of each triangle be without the other triangle.

Because  $AC$  is equal to  $AD$ ,

[*Hypothesis.*

the angle  $ACD$  is equal to the angle  $ADC$ .

[I. 5.

But the angle  $ACD$  is greater than the angle  $BCD$ ,

[*Ax.* 8.

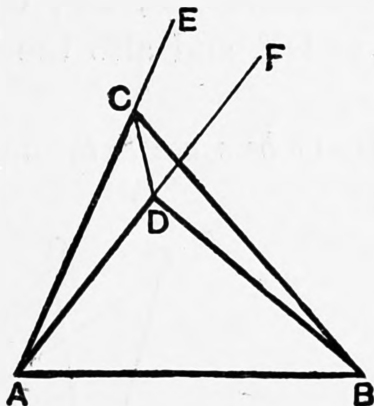
therefore the angle  $ADC$  is also greater than the angle  $BCD$ ;

much more then is the angle  $BDC$  greater than the angle  $BCD$ .



Again, because  $BC$  is equal to  $BD$ , [Hypothesis.  
the angle  $BDC$  is equal to the angle  $BCD$ . [I. 5.  
But it has been shewn to be greater, which is impossible.

CASE II. Let one of the vertices as  $D$  be within the other triangle  $ACB$ , and produce  $AC$ ,  $AD$  to  $E$ ,  $F$ .



Then because  $AC$  is equal to  $AD$ , in the triangle  $ACD$ , [Hyp.  
the angles  $ECD$ ,  $FDC$ , on the other side of the base  $CD$ , are  
equal to one another. [I. 5.

But the angle  $ECD$  is greater than the angle  $BCD$ , [Axiom 8.  
therefore the angle  $FDC$  is also greater than the angle  $BCD$ ;  
much more then is the angle  $BDC$  greater than the angle  $BCD$ .

Again, because  $BC$  is equal to  $BD$ , [Hypothesis.  
the angle  $BDC$  is equal to the angle  $BCD$ . [I. 5.

But it has been shewn to be greater, which is impossible.

CASE III. The case in which the vertex  $D$  of one triangle  
is on a side,  $BC$ , of the other needs no demonstration.

Wherefore, on the same base, etc.

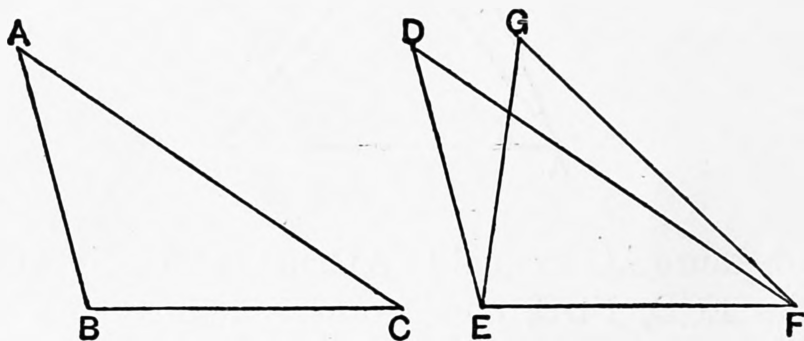
[Q. E. D.]

## PROPOSITION 8. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.*

Let  $ABC$ ,  $DEF$  be two triangles, having the two sides  $AB$ ,  $AC$  equal to the two sides  $DE$ ,  $DF$ , each to each, namely,  $AB$  to  $DE$ , and  $AC$  to  $DF$ , and also the base  $BC$  equal to the base  $EF$  :

*then shall the angle  $BAC$  be equal to the angle  $EDF$ .*



**Proof.** For if the triangle  $ABC$  be applied to the triangle  $DEF$ , so that the point  $B$  may be on the point  $E$ , and the straight line  $BC$  on the straight line  $EF$ , the point  $C$  will also coincide with the point  $F$ ,

because  $BC$  is equal to  $EF$ . [Hypothesis.]

Therefore,  $BC$  coinciding with  $EF$ ,  $BA$  and  $AC$  will coincide with  $ED$  and  $DF$ .

For if not, they must have a different situation as  $EG$ ,  $GF$ ; then on the same base and on the same side of it there would be two triangles having their sides  $DE$ ,  $GE$  equal, and also their sides  $DF$ ,  $GF$  equal.

But this is impossible.

[I 7.]

Therefore the sides  $BA$ ,  $AC$  must coincide with the sides  $ED$ ,  $DF$ .

Therefore also the angle  $BAC$  coincides with the angle  $EDF$ , and is equal to it. [Axiom 9.

Wherefore, *if two triangles, etc.* [Q.E.D.

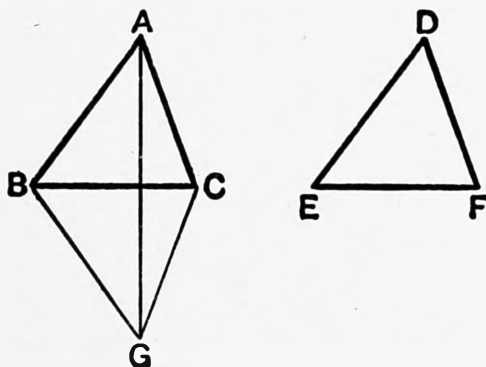
**Cor.** Since the triangle  $ABC$  would coincide with the triangle  $DEF$ , these two triangles *are equal in all respects.*

### ALTERNATIVE PROOF OF PROPOSITION 8.

The following proof is independent of I. 7. It has been recommended by many writers, and is often known as Philo's proof.

Let  $ABC$ ,  $DEF$  be two triangles, having the sides  $AB$ ,  $AC$  equal to the sides  $DE$ ,  $DF$ , each to each, and the base  $BC$  equal to the base  $EF$ :

*the angle  $BAC$  shall be equal to the angle  $EDF$ .*



For, let the triangle  $DEF$  be applied to the triangle  $ABC$ , so that the bases may coincide, the equal sides be conterminous, and the vertices fall on opposite sides of the base.

Let  $GBC$  represent the triangle  $DEF$  thus applied, so that  $G$  corresponds to  $D$ . Join  $AG$ .

Since, by hypothesis,  $BA$  is equal to  $BG$ , the angle  $BAG$  is equal to the angle  $BGA$ . [I. 5.

In the same manner the angle  $CAG$  is equal to the angle  $CGA$ .

Therefore the whole angle  $BAC$  is equal to the whole angle  $BGC$ , that is, to the angle  $EDF$ .

There are two other cases; for the straight line  $AG$  may pass through  $B$  or  $C$ , or it may fall outside  $BC$ : these cases may be treated in the same manner as that which we have considered.

**EXERCISES.**

**\*\*1.** If  $D$  be the middle point of the base  $BC$  of an isosceles triangle  $ABC$ , prove that  $AD$  is perpendicular to  $BC$ .

**\*\*2.** The opposite angles of a rhombus are equal.

**\*\*3.** A diagonal of a rhombus bisects each of the angles through which it passes.

**\*\*4.** The diagonals of a rhombus bisect one another at right angles.

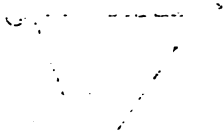
**5.**  $ACB$  and  $ADB$  are two triangles on the same side of  $AB$ , such that  $AC$  is equal to  $BD$ , and  $AD$  is equal to  $BC$ ; if  $AD$  and  $BC$  meet in  $O$ , prove that the triangle  $AOB$  is isosceles.

$ABC$  is an isosceles triangle and  $D, E$  are points in the equal sides  $AB, AC$ , such that  $AD$  and  $AE$  are equal; if  $BE$  and  $CD$  meet in  $F$ , prove that

**6.**  $BCF$  is an isosceles triangle.

**7.**  $AF$  bisects the angle  $BAC$ .

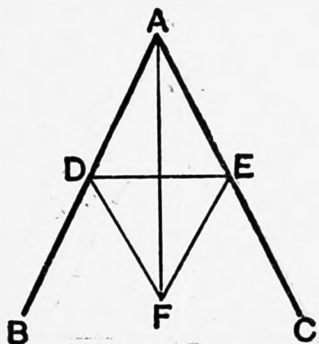
**8.**  $AF$  produced bisects the base  $BC$ .



## PROPOSITION 9. PROBLEM.

*To bisect a given rectilineal angle, that is, to divide it into two equal angles.*

Let  $BAC$  be the given rectilineal angle :  
*it is required to bisect it.*



**Construction.** Take any point  $D$  in  $AB$ , and from  $AC$  cut off  $AE$  equal to  $AD$  ; [I. 3.  
 join  $DE$ , and on  $DE$ , on the side remote from  $A$ , describe the equilateral triangle  $DEF$ . [I. 1.  
 Join  $AF$ .

$AF$  shall bisect the angle  $BAC$ .

**Proof.** In the triangles  $DAF$ ,  $EAF$ ,  
 because  $\begin{cases} AD \text{ is equal to } AE, & [\text{Construction.}] \\ \text{and } AF \text{ is common to both,} \\ \text{and the base } DF \text{ is equal to the base } EF, & [\text{Def. 23.}] \end{cases}$   
 therefore the angle  $DAF$  is equal to the angle  $EAF$ . [I. 8.

Wherefore *the given rectilineal angle  $BAC$  is bisected by the straight line  $AF$ .* [Q. E. F.

*Note.* The equilateral triangle  $DEF$  is to be described on the side remote from  $A$ , because if it were described on the *same* side, its vertex  $F$  might coincide with  $A$  and then the construction would fail.

## EXERCISES.

In the figure of I. 9 prove that

1. the lines  $AF$  and  $DE$  are at right angles.
2. any point  $P$  in  $AF$  is equally distant from the points  $D$  and  $E$ .
3. Divide any angle into four equal parts.



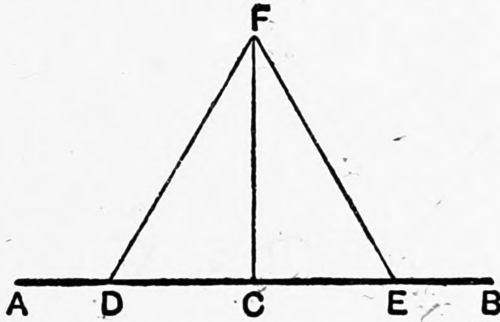


## PROPOSITION 11. PROBLEM.

*To draw a straight line at right angles to a given straight line, from a given point in the same.*

Let  $AB$  be the given straight line, and  $C$  the given point in it:

*it is required to draw from the point  $C$  a straight line at right angles to  $AB$ .*



**Construction.** Take any point  $D$  in  $AC$ , and make  $CE$  equal to  $CD$ . [I. 3.]

On  $DE$  describe the equilateral triangle  $DFE$ , [I. 1.]  
and join  $CF$ .

The straight line  $CF$  shall be the line required

**Proof.** In the triangles  $DCF$ ,  $ECF$ ,  
because  $\begin{cases} DC \text{ is equal to } CE, \\ \text{and } CF \text{ is common to both,} \\ \text{and the base } DF \text{ is equal to the base } EF, \end{cases}$  [Construction.  
[Def. 23.]  
therefore the angle  $DCF$  is equal to the angle  $ECF$ , [I. 8.]  
and they are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [Definition 10.]  
therefore each of the angles  $DCF$ ,  $ECF$  is a right angle.

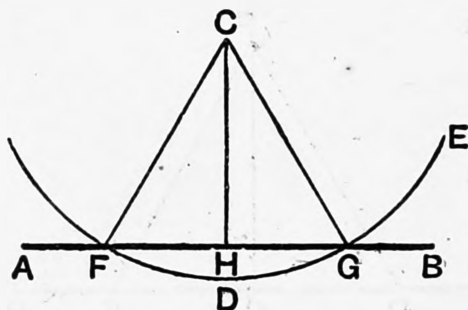
Wherefore, from the given point  $C$  in the given straight line  $AB$ ,  $CF$  has been drawn at right angles to  $AB$ . [Q.E.F.]

## PROPOSITION 12. PROBLEM.

*To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.*

Let  $AB$  be the given straight line, which may be produced to any length both ways, and let  $C$  be the given point without it:

*it is required to draw from the point  $C$  a straight line perpendicular to  $AB$ .*



**Construction.** Take any point  $D$  on the side of  $AB$ , remote from  $C$ , and with centre  $C$  and radius  $CD$  describe the circle  $EGF$ , meeting  $AB$  at  $F$  and  $G$ . [Postulate 3.]

Bisect  $FG$  at  $H$ , [I. 10.]  
and join  $CH$ .

Then  $CH$  shall be the straight line required.

Join  $CF$ ,  $CG$ .

**Proof.** In the triangles  $FHC$ ,  $GHC$ ,

because  $\begin{cases} FH \text{ is equal to } HG, \\ \text{and } HC \text{ is common to both,} \\ \text{and the base } CF \text{ is equal to the base } CG, \end{cases}$  [Construction.]  
therefore the angle  $CHF$  is equal to the angle  $CHG$ , [Definition 15.]  
and they are adjacent angles. [I. 8.]

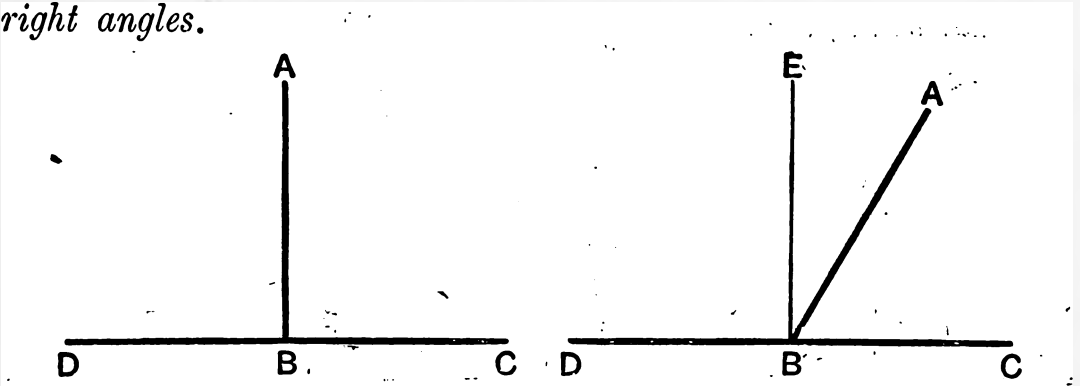
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. [Def. 10.]

Wherefore a perpendicular  $CH$  has been drawn to the given straight line  $AB$  from the given point  $C$  without it. [Q.E.F.]

## PROPOSITION 13. THEOREM.

*The angles which one straight line makes with another straight line on one side of it, are either two right angles, or are together equal to two right angles.*

Let the straight line  $AB$  make with the straight line  $CD$ , on one side of it, the angles  $CBA$ ,  $ABD$  :  
*these shall be either two right angles, or be together equal to two right angles.*



CASE I. If the angle  $CBA$  is equal to the angle  $ABD$ , each of them is a right angle. [Definition 10.]

CASE II. If not, from the point  $B$  draw  $BE$  at right angles to  $CD$ . [I. 11.]

**Proof.** Since the angles  $DBE$ ,  $EBC$  are equal to two right angles, [Constr. and Def. 10.]

and the angle  $EBC$  is equal to the angles  $EBA$ ,  $ABC$ ,  
 therefore the angles  $DBE$ ,  $EBA$ ,  $ABC$  are equal to two right angles. [Axiom 2.]

But the angles  $DBE$ ,  $EBA$  are equal to the angle  $DBA$ ;  
 therefore the angles  $DBA$ ,  $ABC$  are equal to two right angles. [Axiom 2.]

Wherefore, *the angles, etc.* [Q. E. D.]

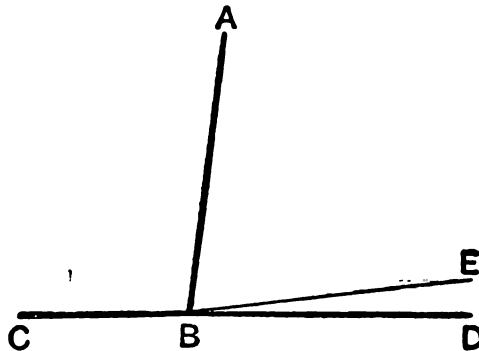
*Note.* Two angles, such as  $ABC$  and  $ABE$ , which together make up a right angle,  $EBC$ , are said to be **complementary**, and each is said to be the **complement** of the other.

Two angles, such as  $ABC$  and  $ABD$ , which together make up two right angles, are said to be **supplementary**, and each is said to be the **supplement** of the other.

## PROPOSITION 14. THEOREM.

*If, at a point in a straight line, two other straight lines, on the opposite sides of it, makes the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.*

At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles :  
BD shall be in the same straight line with CB.



**Proof.** For if BD be not in the same straight line with CB, let BE be in the same straight line with it.

Then because the straight line AB meets the straight line CBE, the angles ABC, ABE are together equal to two right angles. [I. 13.]

But the angles ABC, ABD are also together equal to two right angles. [Hypothesis.]

Therefore the angles ABC, ABE are equal to the angles ABC, ABD. [Axioms 1 and 11.]

From each of these equals take the common angle ABC, and the remaining angle ABE is equal to the remaining angle ABD, [Axiom 3.]

the less to the greater, which is impossible.

Therefore BE is not in the same straight line with CB.

And in the same manner it may be shewn that no other can be in the same straight line with it but BD ; therefore BD is in the same straight line with CB.

Wherefore, *if at a point, etc.*

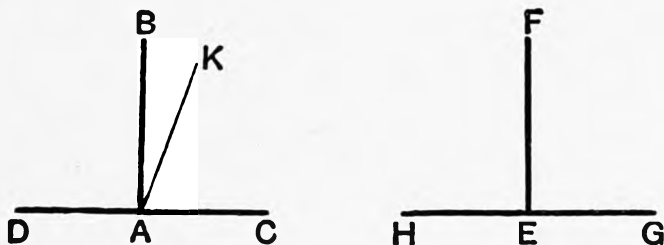
[Q.E.D.]

## NOTE TO I. 14.

Axiom 11, which is first used in this proposition, may be proved as follows :

Let  $AB$  be at right angles to  $DAC$  at the point  $A$ , and  $EF$  at right angles to  $HEG$  at the point  $E$  :

*then shall the angles  $BAC$  and  $FEG$  be equal.*



Take any length  $AC$ , and make  $AD$ ,  $EH$ ,  $EG$  all equal to  $AC$ . Now apply  $HEG$  to  $DAC$ , so that  $H$  may be on  $D$ , and  $HG$  on  $DC$ , and  $B$  and  $F$  on the same side of  $DC$ ; then  $G$  will coincide with  $C$ , and  $E$  with  $A$ .

Also  $EF$  shall coincide with  $AB$ ; for if not, suppose, if possible, that it takes a different position as  $AK$ .

Then the angle  $DAK$  is equal to the angle  $HEF$ , and the angle  $CAK$  to the angle  $GEF$ ;

but the angles  $HEF$  and  $GEF$  are equal; [Hyp. and Def. 10.]

therefore the angles  $DAK$  and  $CAK$  are equal.

But the angles  $DAB$  and  $CAB$  are also equal, [Hyp. and Def. 10.]

and the angle  $CAB$  is greater than the angle  $CAK$ ;

therefore the angle  $DAB$  is greater than the angle  $CAK$ .

Much more then is the angle  $DAK$  greater than the angle  $CAK$ .

But the angle  $DAK$  was shewn to be equal to the angle  $CAK$ ; which is absurd.

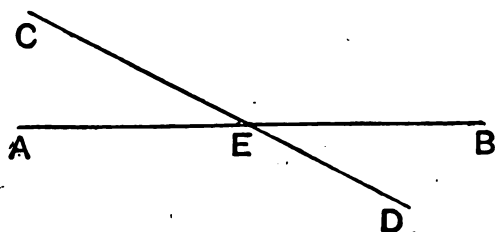
Therefore  $EF$  must coincide with  $AB$ ; and therefore the angle  $FEG$  coincides with the angle  $BAC$ , and is equal to it.

## PROPOSITION 15. THEOREM.

*If two straight lines cut one another, the vertical, or opposite, angles shall be equal.*

Let the two straight lines AB, CD cut one another at the point E:

*the angle AEC shall be equal to the angle DEB,  
and the angle CEB to the angle AED.*



**Proof.** Because the straight line AE meets the straight line CD, the angles CEA, AED are together equal to two right angles. [I. 13.]

Again, because DE meets AB, the angles AED, DEB are also together equal to two right angles. [I. 13.]

Therefore the angles CEA, AED are equal to the angles AED, DEB. [Axioms 1 and 11.]

From each of these equals take the common angle AED; and the remaining angle CEA is equal to the remaining angle DEB. [Axiom 3.]

In the same manner it may be shewn that the angle CEB is equal to the angle AED.

Wherefore, *if two straight lines, etc.*

[Q. E. D.]

**Corollary 1.** From this it is clear that, if two straight lines cut one another, the angles which they make at the point where they cut are together equal to four right angles.

For the angles AEC, AED equal two right angles, and also the angles CEB, BED equal two right angles.

**Corollary 2.** It follows that all the angles made by any number of straight lines meeting at one point are together equal to four right angles.



**EXERCISES ON PROPOSITION 12.**

**1.** Prove that every point in CH, or CH produced, is equidistant from F and G.

**2.** Find a point in a given straight line such that its distances from two given points may be equal.

**3.** Find a point that shall be equidistant from three given points.

**4.** Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by that given straight line.

[Let A, B be the two points and KL the given straight line. Draw AM perpendicular to KL and produce to D, so that MD equals AM. Let DB meet KL in E; then AE, EB are the required lines.]

**5.** D is a given point in the base BC of a triangle ABC; find a straight line such that, if the triangle be folded along it, then the point A will coincide with D.

[The required straight line bisects AD at right angles.]

**EXERCISES ON PROPOSITIONS 13-15.**

**1.** If a right-angled triangle have one of its acute angles double the other, the hypotenuse is double the shorter side.

[If ABC be the right-angled triangle having the angle C double the angle B, produce CA to D, where  $CA = AD$ , and prove that CBD is equilateral, etc.]

**2.** If two isosceles triangles are on the same base, the straight line joining their vertices, or that straight line produced, will bisect the base at right angles.

**3.** A given angle BAC is bisected; if CA is produced to G and the adjacent angle BAG bisected, the two bisecting lines are at right angles. [These two bisecting lines are called the interior and the exterior bisectors of the angles BAC.]

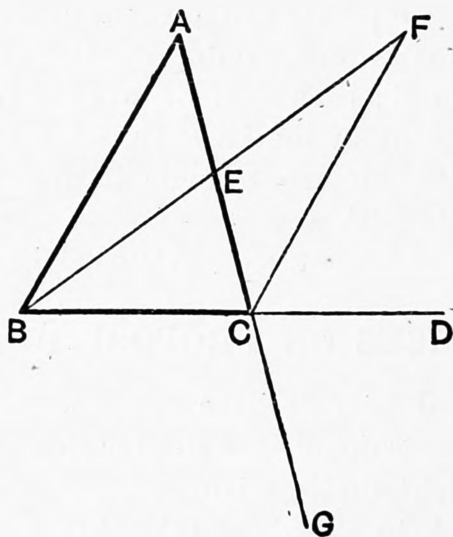
**\*\*4.** If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

## PROPOSITION 16. THEOREM.

*If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.*

Let  $ABC$  be a triangle, and let one side  $BC$  be produced to  $D$ :

*the exterior angle  $ACD$  shall be greater than either of the interior opposite angles  $CBA$ ,  $BAC$ .*



**Construction.** Bisect  $AC$  at  $E$ , [I. 10.  
join  $BE$  and produce it to  $F$ , making  $EF$  equal to  $EB$ , [I. 3.  
and join  $FC$ .

**Proof.** In the triangles  $AEB$ ,  $CEF$ ,  
because  $\begin{cases} AE = EC, & \text{[Construction.]} \\ \text{and } EB = EF, & \text{[Construction.]} \\ \text{and the angle } AEB = \text{the angle } CEF, \end{cases}$   
since they are opposite vertical angles; [I. 15.  
therefore the angle  $BAE = \text{the angle } ECF$ . [I. 4.  
But the angle  $ECD$  is greater than the angle  $ECF$ . [Axiom 8.  
Therefore the angle  $ACD$  is greater than the angle  $BAE$ .

In the same manner, if  $BC$  be bisected, and the side  $AC$  be produced to  $G$ , it may be shewn that the angle  $BCG$ , that is, the angle  $ACD$ , is greater than the angle  $ABC$ . [I. 15.

Wherefore, if one side, etc.

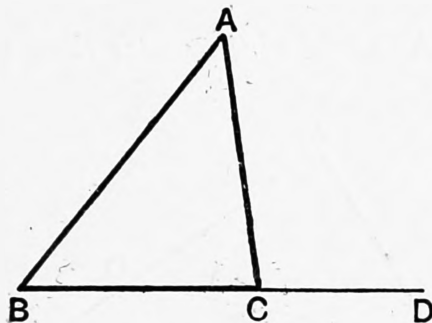
[Q.E.D.]

## PROPOSITION 17. THEOREM.

*Any two angles of a triangle are together less than two right angles.*

Let  $ABC$  be a triangle :

*any two of its angles shall be together less than two right angles.*



**Construction.** Produce  $BC$  to  $D$ .

**Proof.** Because  $ACD$  is the exterior angle of the triangle  $ABC$ , it is greater than the interior opposite angle  $ABC$ . [I. 16. To each of these add the angle  $ACB$ .

Therefore the angles  $ACD$ ,  $ACB$  are greater than the angles  $ABC$ ,  $ACB$ .

But the angles  $ACD$ ,  $ACB$  together = two right angles. [I. 13. Therefore the angles  $ABC$ ,  $ACB$  are together less than two right angles.

In the same manner it may be shewn that the angles  $BAC$ ,  $ACB$ , as also the angles  $CAB$ ,  $ABC$ , are together less than two right angles.

Wherefore, *any two angles, etc.*

[Q.E.D.]

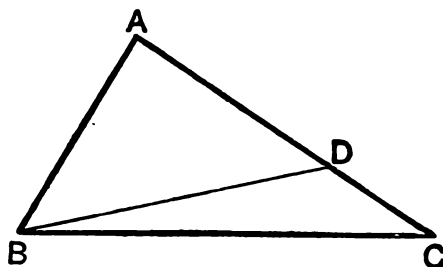
**Note.** Every triangle has at least two acute angles ; for if a triangle had two obtuse angles, their sum would not be less than two right angles, and this is impossible by the foregoing proposition.

## PROPOSITION 18. THEOREM.

*If one side of a triangle be greater than a second side, the angle opposite the first side shall be greater than the angle opposite the second.*

Let  $ABC$  be a triangle, of which the side  $AC$  is greater than the side  $AB$ :

*then shall the angle  $ABC$  be also greater than the angle  $ACB$ .*



**Construction.** Because  $AC$  is greater than  $AB$ ,  
cut off  $AD$  equal to  $AB$ , [I. 3.  
and join  $BD$ .

**Proof.** Because  $ADB$  is the exterior angle of the triangle  $BDC$ , it is greater than the interior opposite angle  $DCB$ . [I. 16.

But the angle  $ADB =$  the angle  $ABD$ , [I. 5.

because the side  $AD =$  the side  $AB$ . [Construction.

Therefore the angle  $ABD$  is also greater than the angle  $ACB$ .

Much more then is the angle  $ABC$  greater the angle  $ACB$ .

[Axiom 8.

Wherefore, *the greater side, etc.*

[Q.E.D.

## EXERCISE.

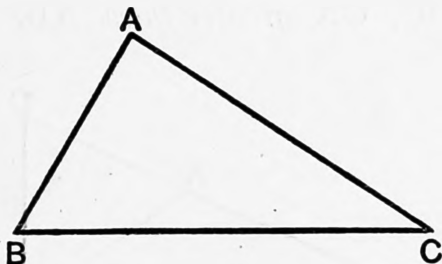
$ABCD$  is a quadrilateral of which  $AD$  is the longest side and  $BC$  the shortest; shew that the angle  $ABC$  is greater than the angle  $ADC$ , and the angle  $BCD$  greater than the angle  $BAD$ .

## PROPOSITION 19. THEOREM.

*If one angle of a triangle be greater than a second angle, the side opposite the first angle shall be greater than the side opposite the second.*

Let  $ABC$  be a triangle, of which the angle  $ABC$  is greater than the angle  $ACB$  :

*then shall the side  $AC$  be also greater than the side  $AB$ .*



**Proof.** For if not,  $AC$  must be either equal to  $AB$  or less than  $AB$ .

But  $AC$  is not equal to  $AB$ ,  
for then the angle  $ABC$  would be equal to the angle  $ACB$  ; [I. 5.  
but it is not ; [Hypothesis.  
therefore  $AC$  is not equal to  $AB$ .

Neither is  $AC$  less than  $AB$ ,  
for then the angle  $ABC$  would be less than the angle  $ACB$  ; [I. 18.  
but it is not ; [Hypothesis.  
therefore  $AC$  is not less than  $AB$ .

And it has been shewn that  $AC$  is not equal to  $AB$ .

Therefore  $AC$  is greater than  $AB$ .

Wherefore, *the greater angle, etc.*

[Q. E. D.

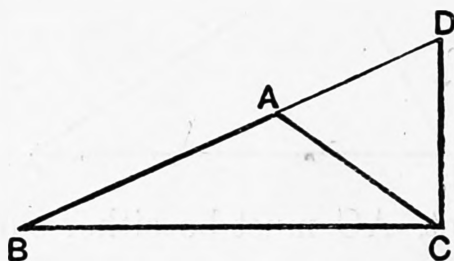
*Note 1.* In order to assist the student in remembering which of the two foregoing propositions is proved directly and which indirectly, it may be observed that the order is similar to that of Propositions 5 and 6.

*Note 2.* Simson's enunciations of Propositions 18 and 19 were :  
*the greater side of any triangle has the greater angle opposite it ; and  
the greater angle of any triangle is subtended by the greater side, that is,  
has the greater side opposite to it.*

## PROPOSITION 20. THEOREM.

*Any two sides of a triangle are together greater than the third side.*

Let  $ABC$  be a triangle :  
*any two sides of it are together greater than the third side,*  
*namely,  $BA, AC$  greater than  $BC$  ;*  
 *$AB, BC$  greater than  $AC$  ;*  
*and  $BC, CA$  greater than  $AB$ .*



**Construction.** Produce  $BA$  to  $D$ ,  
 making  $AD$  equal to  $AC$ , [I. 3.]  
 and join  $DC$ .

**Proof.** Because  $AD = AC$ , [Construction.]  
 the angle  $ADC =$  the angle  $ACD$ . [I. 5.]

But the angle  $BCD$  is greater than the angle  $ACD$ . [Ax. 8.]

Therefore the angle  $BCD$  is greater than the angle  $BDC$ .

And because the angle  $BCD$  of the triangle  $BCD$  is greater than its angle  $BDC$ ,

therefore the side  $BD$  is greater than the side  $BC$ . [I. 19.]

But  $BD$  is equal to  $BA$  and  $AC$ .

Therefore  $BA$  and  $AC$  are together greater than  $BC$ .

In the same manner it may be shewn that

$AB, BC$  are together greater than  $AC$ ,

and  $BC, CA$  together greater than  $AB$ .

Wherefore, *any two sides, etc.*

[Q.E.D.]

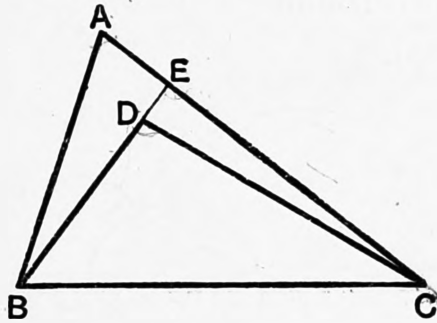


## PROPOSITION 21. THEOREM.

*If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

Let  $ABC$  be a triangle, and from  $B$  and  $C$ , the ends of the side  $BC$ , let the two straight lines  $BD$ ,  $CD$  be drawn to any point  $D$  within the triangle :

*$BD$ ,  $DC$  shall be less than the other two sides  $BA$ ,  $AC$  of the triangle, but shall contain an angle  $BDC$  greater than the angle  $BAC$ .*



**Construction.** Produce  $BD$  to meet  $AC$  at  $E$ .

**Proof.** In the triangle  $ABE$  the two sides  $BA$ ,  $AE$  are together greater than the side  $BE$  ; [I. 20.

to each of these add  $EC$  ;

$\therefore BA$ ,  $AC$  are greater than  $BE$ ,  $EC$ .

Again, the two sides  $DE$ ,  $EC$  of the triangle  $DEC$  are greater than the third side  $DC$  ; [I. 20.

to each of these add  $BD$ .

$\therefore BE$ ,  $EC$  are greater than  $BD$ ,  $DC$ .

But it has been shewn that  $BA$ ,  $AC$  are greater than  $BE$ ,  $EC$  ; much more then are  $BA$ ,  $AC$  greater than  $BD$ ,  $DC$ .

Again, the exterior angle  $BDC$  of the triangle  $CDE$  is greater than the interior opposite angle  $CEB$ . [I. 16.

For the same reason, the exterior angle  $CEB$  of the triangle  $ABE$  is greater than the angle  $BAE$  ;

much more then is the angle  $BDC$  greater than the angle  $BAE$ , that is,  $BAC$ .

Wherefore, *if from the ends, etc.*

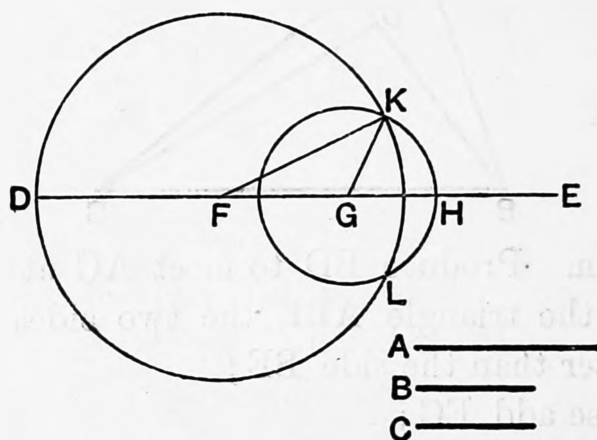
[Q. E. D.]

## PROPOSITION 22. PROBLEM.

*To make a triangle of which the sides shall be equal to three given straight lines, any two whatever of which are together greater than the third.*

Let  $A, B, C$  be the three given straight lines, of which any two whatever are greater than the third; namely,  $A$  and  $B$  together greater than  $C$ ;  $A$  and  $C$  together greater than  $B$ ; and  $B$  and  $C$  together greater than  $A$ :

*it is required to make a triangle of which the sides shall be equal to  $A, B, C$ , each to each.*



**Construction.** Take a straight line  $DE$  terminated at the point  $D$ , but unlimited towards  $E$ , and make  $DF$  equal to  $A$ ,  $FG$  equal to  $B$ , and  $GH$  equal to  $C$ . [I. 3.]

With centre  $F$ , and radius  $FD$ , describe the circle  $DKL$ .

[Postulate 3.]

With centre  $G$ , and radius  $GH$ , describe the circle  $HKL$ , and let it cut the former circle at  $K$ .

Join  $KF, KG$ .

The triangle  $KFG$  shall be drawn as required.

**Proof.** Because the point  $F$  is the centre of the circle  $DKL$ ,

$FD = FK$ . [Definition 15.]

But  $FD = A$ ; [Construction.]

$\therefore FK = A$ . [Axiom 1.]

Again, because the point  $G$  is the centre of the circle  $HLK$ ,  
 $GH = GK$ . [Definition 15.  
 But  $GH = C$ ; [Construction.  
 $\therefore GK = C$ . [Axiom 1.  
 Also  $FG = B$ . [Construction.

Therefore the three straight lines  $KF$ ,  $FG$ ,  $GK$  are equal to the three  $A$ ,  $B$ ,  $C$ .

Wherefore the triangle  $KFG$  has its three sides  $KF$ ,  $FG$ ,  $GK$  equal to the three given straight lines  $A$ ,  $B$ ,  $C$ . [Q.E.F.

### EXERCISES ON PROPOSITION 19.

1.  $ABC$  is a triangle and the angle  $A$  is bisected by a straight line which meets  $BC$  at  $D$ ; shew that  $BA$  is greater than  $BD$ , and  $CA$  greater than  $CD$ .

2. If a straight line be drawn through  $A$  one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at  $F$ , shew that  $AF$  is greater than the diagonal of the square.

3. Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either of them if they be equal.

4.  $ABC$  is a triangle in which  $BA$  is greater than  $CA$ ; the angle  $A$  is bisected by a straight line which meets  $BC$  at  $D$ ; shew that  $BD$  is greater than  $CD$ .

[Take a point  $E$  on  $AB$  such that  $AE = AC$ ; then the angle  $AED$  equals the angle  $ACB$ , and  $ED$  is equal to  $CD$ ;  $\therefore$  the angle  $BED$  equals the exterior angle at  $C$ , and is thus greater than the angle  $ABC$ ;  $\therefore BD$  is greater than  $DE$ , that is, than  $CD$ .]

\*\*5. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and, of others, that which is nearer to the perpendicular is less than the more remote; and two, and two only, straight lines, each equal to a given straight line, can be drawn from the given point to the given straight line, one on each side of the perpendicular.

### EXERCISES ON PROPOSITION 20.

\*\*1. The difference between any two sides of a triangle is less than the third side.

\*\*2. The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

**\*\*3.** The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

[*N.B.*—The straight line drawn from any angular point to the middle point of the opposite side is called a **Median**.]

[Let  $D$  be the middle point of the base  $BC$  of the triangle  $ABC$ . Produce  $AD$  to  $E$  where  $DE=AD$ , and join  $BE$ ,  $EC$ ; prove  $EC=AB$ , etc.]

**4.** The sum of the diagonals of a quadrilateral is less than the sum of the straight lines drawn to its angular points from any point except the intersection of its diagonals.

**5.** The four sides of any quadrilateral are together greater than the two diagonals together, but are less than twice the sum of the diagonals.

**6.** If through the ends of the base of a triangle with unequal sides lines be drawn to any point in the bisector of the vertical angle their difference is less than the difference of the sides.

[Let  $ABC$  be the  $\triangle$ ,  $AD$  the bisector of the angle  $A$ , and  $P$  any point on  $AD$ . Take a point  $E$  on  $AB$  such that  $AE=AC$ ; then  $PC=PE$ . Now  $BP < BE$ ,  $EP$ ;

$\therefore BP - EP < BE$ , i.e.  $BP - CP < BA - AE$ , i.e.  $< BA - AC$ .]

**7.** If lines be drawn to any point in the bisector of the exterior angle of a triangle from the ends of its base, their sum is greater than the sum of the sides.

[Take a point  $E$  on  $BA$  produced such that  $AE$  equals  $AC$ , and proceed as in the last Example.]

## EXERCISE ON PROPOSITION 21.

$ABC$  is a triangle, and  $P$  is any point within it: shew that the sum of  $PA$ ,  $PB$ ,  $PC$  is less than the sum of the sides of the triangle.

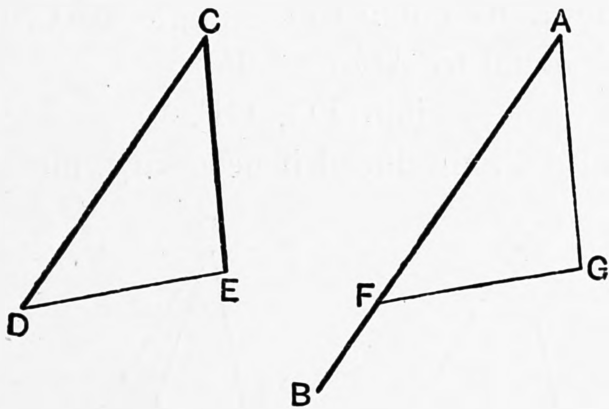
## PROPOSITION 23. PROBLEM.

*At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.*

Let  $AB$  be the given straight line, and  $A$  the given point in it, and  $DCE$  the given rectilineal angle :  
*it is required to make at the given point  $A$ , in the given straight line  $AB$ , an angle equal to the given rectilineal angle  $DCE$ .*

**Construction.** In  $CD$ ,  $CE$  take any points  $D$ ,  $E$ , and join  $DE$ .

From  $AB$  cut off  $AF$  equal to  $CD$ , and construct the triangle  $AFG$  so that the sides  $AF$ ,  $FG$ ,  $GA$  may be respectively equal to the sides  $CD$ ,  $DE$ ,  $EC$ . [I. 22.]



The angle  $FAG$  shall be equal to the angle  $DCE$ .

**Proof.** In the triangles  $DCE$ ,  $FAG$ ,

because  $\begin{cases} FA = DC, \\ \text{and } AG = CE, \\ \text{and the base } FG = \text{the base } DE; \end{cases}$  [Construction.]  
 $\therefore$  the angle  $FAG = \text{the angle } DCE$ . [I. 8.]

Wherefore, *at the given point  $A$ , in the given straight line  $AB$ , the angle  $FAG$  has been made equal to the given rectilineal angle  $DCE$ .* [Q.E.F.]

[For Exercises see page 43.]

## PROPOSITION 24. THEOREM.

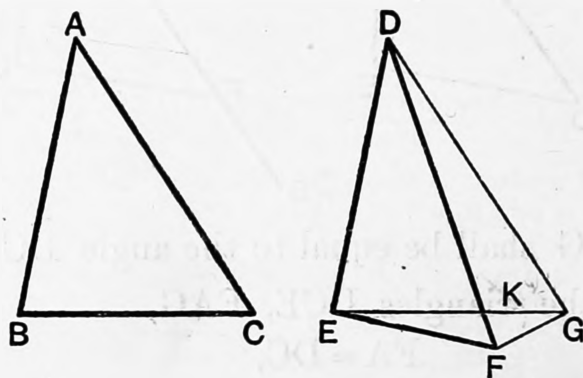
*If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.*

Let  $ABC$ ,  $DEF$  be two triangles, which have the side  $AB$  equal to the side  $DE$ , the side  $AC$  equal to the side  $DF$ , but the angle  $BAC$  greater than the angle  $EDF$  :  
*the base  $BC$  shall be greater than the base  $EF$ .*

Of the two sides  $DE$ ,  $DF$ , let  $DE$  be the side which is not greater than the other.

**Construction.** At the point  $D$  in the straight line  $DE$ , make the angle  $EDG$  equal to the angle  $BAC$ , [I. 23.  
 and make  $DG$  equal to  $AC$  or  $DF$ ; [I. 3.  
 join  $EG$ ,  $GF$ .

Let  $EG$  and  $DF$ , produced if necessary, meet in  $K$ .



**Proof.** Since  $DE$  is not greater than  $DF$ , that is,  $DG$ , that is, since  $DE$  is equal to, or less than,  $DG$ , therefore the angle  $DGE$  is equal to, or less than, the angle  $DEG$ . [I. 5, 18.

But the angle  $DKG$  is greater than the angle  $DEG$ , [I. 16.  
 therefore the angle  $DKG$  is greater than the angle  $DGE$ ;  
 therefore the side  $DG$  is greater than  $DK$ ,  
 that is,  $DF$  is greater than  $DK$ .

In the triangles  $ABC$ ,  $DEG$ ,

because  $\left\{ \begin{array}{l} AB = DE, \\ \text{and } AC = DG, \\ \text{and the angle } BAC = \text{the angle } EDG, \end{array} \right. \begin{array}{l} [\textit{Hypothesis.}] \\ [\textit{Construction.}] \\ [\textit{Construction.}] \end{array}$

$\therefore$  the base  $BC =$  the base  $EG$ . [I. 4.]

And because  $DG = DF$ , [Construction.]

the angle  $DGF =$  the angle  $DFG$ . [I. 5.]

But the angle  $DGF$  is greater than the angle  $EGF$ ; [Axiom 8.]

$\therefore$  the angle  $DFG$  is greater than the angle  $EGF$ .

Much more then is the angle  $EFG$  greater than the angle  $EGF$ . [Axiom 8.]

And because the angle  $EFG$  of the triangle  $EFG$  is greater than its angle  $EGF$ ,

$\therefore$  the side  $EG$  is greater than the side  $EF$ . [I. 19.]

But  $EG$  was shown to be equal to  $BC$ ;

$\therefore$   $BC$  is greater than  $EF$ .

Wherefore, *if two triangles, etc.*

[Q.E.D.]

### EXERCISES ON PROPOSITION 23.

1. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

2. If the angle  $C$  of a triangle is equal to the sum of the angles  $A$  and  $B$ , the side  $AB$  is equal to twice the straight line joining  $C$  to the middle point of  $AB$ .

3. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

[If  $AB$  be the given base, make the angle  $BAD$  equal to the given  $\angle$  and  $AD$  equal to the given sum of the sides; at  $B$ , on same side of  $BD$  as  $A$ , make  $\angle DBK = \angle ADB$ ; let  $BK$  meet  $AD$  in  $K$ . Then  $KAB$  is the required  $\triangle$ .]

4. Construct a triangle having given the base, one of the base angles, and the difference of the sides.

[If  $AB$  be the given base, make the angle  $BAD$  equal to the given  $\angle$  and  $AD$  equal to the given difference; at  $B$ , on the other side of  $BD$  from  $A$ , make the  $\angle DBK =$  exterior  $\angle$  of  $ADB$ ; if  $BK$  meet  $AD$  produced in  $K$ , then  $KAB$  is the required  $\triangle$ .]

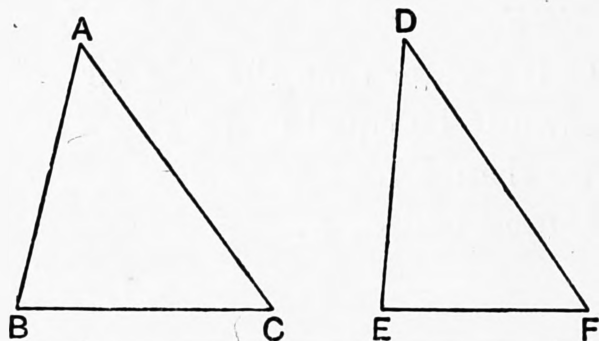
5.  $A$  is a given point, and  $B$  is a given point in a given straight line: it is required to draw from  $A$  to the given straight line, a straight line  $AP$ , such that the sum of  $AP$  and  $PB$  may be equal to a given length.



## PROPOSITION 25. THEOREM.

*If two triangles have two sides of the one respectively equal to two sides of the other, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the corresponding sides of the other.*

Let  $ABC$ ,  $DEF$  be two triangles, which have the side  $AB$  equal to the side  $DE$ , the side  $AC$  equal to the side  $DF$ , but the base  $BC$  greater than the base  $EF$  :  
*the angle  $BAC$  shall be greater than the angle  $EDF$ .*



**Proof.** For if not, the angle  $BAC$  must be either equal to the angle  $EDF$ , or less than, the angle  $EDF$ .

Now the angle  $BAC$  does not = the angle  $EDF$ , for then

the base  $BC$  would = the base  $EF$ ; [I. 4.

but it does not; [*Hypothesis.*

$\therefore$  the angle  $BAC$  does not = the angle  $EDF$ .

Neither is the angle  $BAC$  less than the angle  $EDF$ ,

for then the base  $BC$  would be less than the base  $EF$ ; [I. 24.

but it is not; [*Hypothesis.*

$\therefore$  the angle  $BAC$  is not less than the angle  $EDF$ .

$\therefore$  the angle  $BAC$  is not equal, or less than, the angle  $EDF$ .

$\therefore$  the angle  $BAC$  must be greater than the angle  $EDF$ .

Wherefore, *if two triangles, etc.*

[Q.E.D.]

*Note.* Proposition 25, as well as Proposition 19, are proved by the method of "Exhaustion," that is, by shewing that no other conclusion, *except the given one*, can follow.

### EXERCISES.

1. ABCD is a quadrilateral having AB, CD equal, but the diagonal BD greater than the diagonal AC ; prove that the angle DCB is greater than the angle ABC.
2. ABC is a triangle having AB less than AC, and G is any point in the straight line joining A to the middle point of BC ; prove that G is nearer to B than to C.

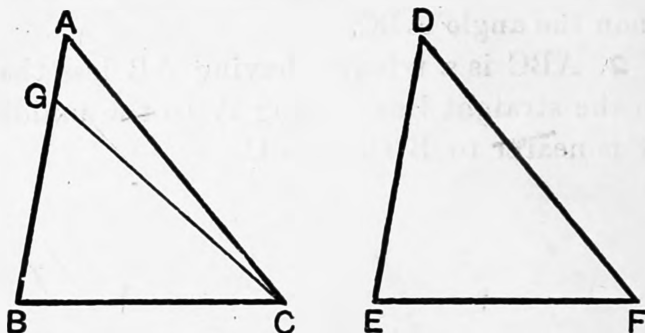
## PROPOSITION 26. THEOREM.

*If two triangles have two angles of the one respectively equal to two angles of the other, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the triangles be equal in all respects, these sides being equal which are opposite to equal angles.*

CASE I. When the equal sides are adjacent to the equal angles.

Let  $ABC$ ,  $DEF$  be two triangles. which have the angle  $ABC$  equal to  $DEF$ , and the angle  $BCA$  equal to  $EFD$ , and the side  $BC$  equal to the side  $EF$ ,  
*then shall the triangles be equal in all respects, so that  $AB = DE$ ,  $AC = DF$ , and the angle  $BAC =$  the angle  $EDF$ .*

For, if  $AB$  be not equal to  $DE$ , one of them must be greater than the other. Let  $AB$  be the greater, and make  $BG$  equal to  $ED$ ,  
 [I. 3.  
 and join  $GC$ .



Then, in the triangles  $GBC$ ,  $DEF$ ,

because  $\begin{cases} GB = DE, & \text{[Construction.]} \\ \text{and } BC = EF, & \\ \text{and the angle } GBC = \text{the angle } DEF, & \text{[Hypothesis.]} \end{cases}$

$\therefore$  the triangles are equal in all respects,

and the angle  $GCB =$  the angle  $DFE$ . [I. 4.]

But the angle  $DFE =$  the angle  $ACB$ . [Hypothesis.]

$\therefore$  the angle  $GCB =$  the angle  $ACB$ , [Axiom 1.]

the less to the greater, which is impossible.

Therefore  $AB$  is not unequal to  $DE$ , that is, it is equal to it.

Then, in the triangles  $ABC$ ,  $DEF$ ,

because  $\begin{cases} AB = DE, & \text{[Proved.]} \\ \text{and } BC = EF, & \text{[Hypothesis.]} \\ \text{and the angle } ABC = \text{the angle } DEF; & \text{[Hypothesis.]} \end{cases}$

$\therefore$  the base  $AC =$  the base  $DF$ ;

and the third angle  $BAC =$  the third angle  $EDF$ . [I. 4.]

CASE II. When the equal sides are opposite to equal angles in each triangle.

Let  $ABC$ ,  $DEF$  be two triangles having the two angles  $ABC$ ,  $ACB$  respectively equal to the angles  $DEF$ ,  $DFE$ , and the side  $AB$  equal to  $DE$ ,

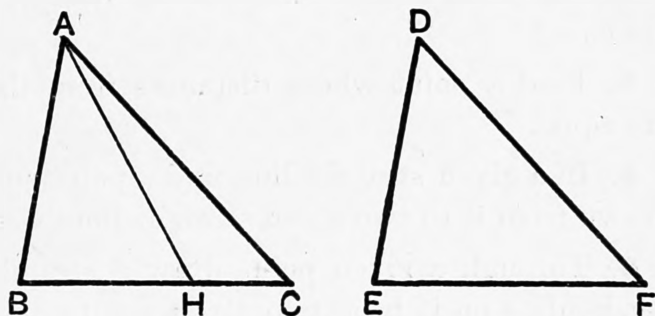
*then the triangles shall be equal in all respects, so that  $BC = EF$ ,  $AC = DF$ , and also the third angle  $BAC =$  the third angle  $EDF$ .*

For if  $BC$  be not equal to  $EF$ , one of them must be greater than the other.

Let  $BC$  be the greater, and make  $BH$  equal to  $EF$ ,

[I. 3.

and join  $AH$ .



Then, in the triangles  $ABC$ ,  $DEF$ ,

because  $\left\{ \begin{array}{l} AB = DE, \\ \text{and } BH = EF, \\ \text{and the angle } ABH = \text{the angle } DEF; \end{array} \right. \begin{array}{l} \text{[Hypothesis.]} \\ \text{[Construction.]} \\ \text{[Hypothesis.]} \end{array}$

$\therefore$  the triangle  $ABH =$  the triangle  $DEF$  in all respects,

and therefore the angle  $BHA =$  the angle  $EFD$ . [I. 4.

But the angle  $EFD =$  the angle  $BCA$ ; [Hypothesis.

$\therefore$  the angle  $BHA =$  the angle  $BCA$ , [Axiom 1.

that is, the exterior angle  $BHA$  of the triangle  $AHC$  is equal to its interior opposite angle  $BCA$ , which is impossible. [I. 16.

Therefore  $BC$  is not unequal to  $EF$ , that is, it is equal to it.

Hence in the triangles  $ABC$ ,  $DEF$ ,

because  $\left\{ \begin{array}{l} AB = DE, \\ \text{and } BC = EF, \\ \text{and the angle } ABC = \text{the angle } DEF, \end{array} \right. \begin{array}{l} \text{[Hypothesis.]} \\ \text{[Proved.]} \end{array}$

$\therefore$  the base  $AC =$  the base  $DF$ ,

and the third angle  $BAC =$  the third angle  $EDF$ . [I. 4.

Wherefore, *if two triangles, etc.*

[Q. E. D.

**EXERCISES ON PROPOSITION 26.**

**\*\*1.** The perpendiculars from the ends of the base of an isosceles triangle upon the opposite sides are equal.

**\*\*2.** The perpendicular from the vertex on the base of an isosceles triangle bisects both the base and the vertical angle.

**\*\*3.** The perpendiculars let fall on two given straight lines  $AB$ ,  $AC$  from any point in the straight line bisecting the angle between them are equal.

**4.** Find a point whose distances from the sides of a given triangle are equal.

**5.** In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines which intersect shall be equal.

**6.** Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

**7.** A straight line bisects the angle  $A$  of a triangle  $ABC$ ; from  $B$  a perpendicular is drawn to this bisecting straight line, meeting it at  $D$ , and  $BD$  is produced to meet  $AC$  or  $AC$  produced at  $E$ : shew that  $BD$  is equal to  $DE$ .

**8.**  $AB$ ,  $AC$  are any two straight lines meeting at  $A$ : through any point  $P$  draw a straight line meeting them at  $E$  and  $F$ , such that  $AE$  may be equal to  $AF$ .

**9.** If the diagonal  $AC$  of a quadrilateral  $ABCD$  bisect the angles at  $A$  and  $C$ , prove that it is at right angles to the other diagonal  $BD$ .

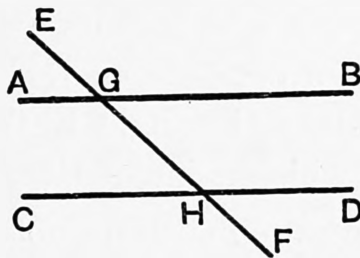
**10.** If the sides  $AB$ ,  $AC$  of a triangle  $ABC$  be produced to  $D$  and  $E$  and the bisectors of the angles  $BCE$ ,  $CBD$  meet in  $O$ , prove that the perpendiculars from  $O$  upon  $AD$ ,  $AE$ , and  $BC$  are equal.

*Note to I. 26.* The first twenty-six Propositions constitute a distinct section of the first Book of the *Elements*. The principal results are those contained in Propositions 4, 8, 26; in each of these Propositions it is shewn that two triangles which agree in three respects agree entirely. For another case in which two triangles are equal in all respects (in addition to the cases considered in I. 4, 8, 26) the student may refer to Page 322.

The Propositions from I. 27 to I. 34 inclusive may be said to constitute the second section of the first Book. They relate to the theory of parallel straight lines.

### ON THE ANGLES MADE BY ONE STRAIGHT LINE WITH TWO OTHER STRAIGHT LINES.

When two straight lines AB and CD are met by a third straight line EF in two points G, H, names for the sake of distinction are given to the angles at G and H.



Thus EGB, AGE, CHF and FHD are called **exterior** angles ;  
BGH, AGH, CHG, GHD are called **interior** angles ;  
BGH and GHC are called **alternate** angles,  
and so are also AGH and GHD ;  
also GHD is said to be *the interior and opposite angle of the exterior angle EGB*.

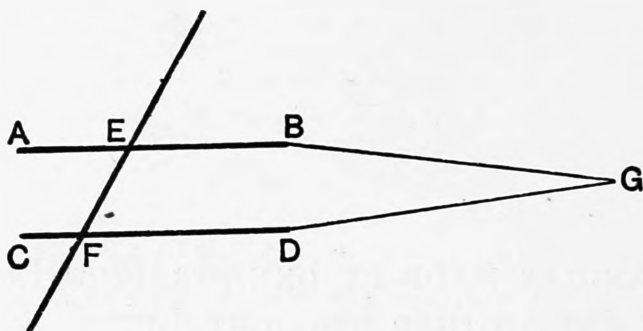
The words *interior*, *exterior*, and *alternate* are often abbreviated into *int.*, *ext.*, and *alt.*

## PROPOSITION 27. THEOREM.

*If a straight line, meeting two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.*

Let the straight line EF, which meets the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another :

*AB shall be parallel to CD.*



**Proof.** For if not, AB and CD, being produced, will meet either towards B, D or towards A, C. Let them be produced and meet towards B, D at the point G.

Therefore GEF is a triangle, and its exterior angle AEF is greater than the interior opposite angle EFG. [I. 16.]

But the angle AEF also = the angle EFG; [Hypothesis.  
which is impossible.

∴ AB and CD, being produced, do not meet towards B, D.

In the same manner it may be shewn that they do not meet towards A, C.

∴ AB is parallel to CD. [Definition 29.]

Wherefore, *if a straight line, etc.*

[Q. E. D.]



## PROPOSITION 28. THEOREM.

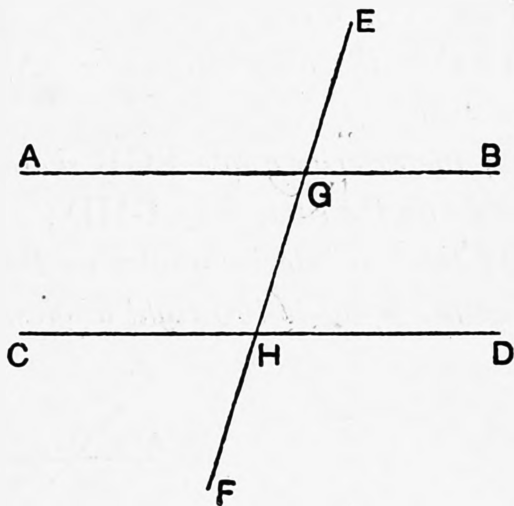
*If a straight line, meeting two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or if it make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.*

Let the straight line EF,  
which meets the two straight  
lines AB, CD,

(1) make the exterior angle  
EGB equal to the interior and  
opposite angle GHD on the  
same side ;

or (2) make the interior angles  
on the same side BGH, GHD  
together equal to two right  
angles :

AB shall be parallel to CD.



**Proof.** (1) Because the angle EGB = the angle GHD, [*Hyp.*  
and the angle EGB = the angle AGH, [I. 15.

$\therefore$  the angle AGH = the angle GHD, [*Axiom 1.*

and they are alternate ;

$\therefore$  AB is parallel to CD. [I. 27.

(2) Again, because the angles BGH, GHD are together  
equal to two right angles, [*Hypothesis.*

and the angles AGH, BGH together = two right angles, [I. 13.

$\therefore$  the angles AGH, BGH = the angles BGH, GHD. [*Axs. 1, 11.*

Take away the common angle BGH ;

$\therefore$  the angle AGH = the angle GHD, [*Axiom 3.*

and they are alternate ;

$\therefore$  AB is parallel to CD. [I. 27.

Wherefore, if a straight line, etc.

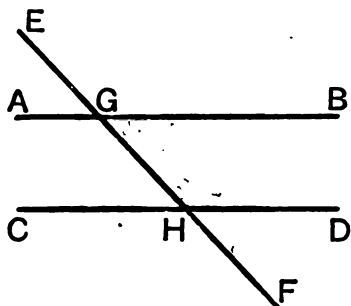
[Q. E. D.

## PROPOSITION 29. THEOREM.

*If a straight line meet two parallel straight lines, it makes*  
 (1) *the alternate angles equal to one another ;*  
 (2) *the exterior angle equal to the interior and opposite angle on the same side ;*  
 and also (3) *the two interior angles on the same side together equal to two right angles.*

Let the straight line EF meet the two parallel straight lines AB, CD :

- (1) *then the alternate angles AGH, GHD shall be equal to one another ;*  
 (2) *the exterior angle EGB shall be equal to the interior and opposite angle on the same side GHD ; and*  
 (3) *the two interior angles on the same side BGH, GHD shall be together equal to two right angles.*



**Proof.** (1) If the angle AGH be not equal to the angle GHD, one of them must be greater than the other ; let the angle AGH be the greater.

Then the angle AGH is greater than the angle GHD ;  
 to each of them add the angle BGH ;

$\therefore$  the angles AGH, BGH are greater than the angles BGH, GHD.

But the angles AGH, BGH together = two right angles ; [I. 13.  
 $\therefore$  the angles BGH, GHD are together less than two right angles ;

$\therefore$  AB, CD, if continually produced, will meet on the side of GH towards B and D.

[Axiom 12.

But they never meet, since they are parallel by hypothesis.

$\therefore$  the angle AGH is not unequal to the angle GHD, that is, it is equal to it.

(2) Again, because the angle AGH = the angle EGB; [I. 15. therefore the angle EGB = the angle GHD. [Axiom 1.

(3) Add to each of these the angle BGH.

$\therefore$  the angles EGB, BGH = the angles BGH, GHD. [Axiom 2.

But the angles EGB, BGH together = two right angles. [I. 13.

$\therefore$  the angles BGH, GHD together = two right angles. [Ax. 1.

Wherefore, *if a straight line, etc.* [Q.E.D.

[For Exercises see page 55.]

### NOTE ON EUCLID'S TWELFTH AXIOM.

In I. 29 Euclid uses for the first time his twelfth axiom. The theory of parallel lines has always been considered the great difficulty of elementary geometry, and many attempts have been made to overcome this difficulty in a better way than Euclid has done. We shall not give an account of these attempts.

Speaking generally, it may be said that the methods which differ substantially from Euclid's involve, in the first place, an axiom as difficult as his, and then an intricate series of propositions; while in Euclid's method, after the axiom is once admitted, the remaining process is simple and clear.

One modification of Euclid's axiom has been proposed, which appears to diminish the difficulty of the subject. This consists in assuming, instead of Euclid's axiom, the following:

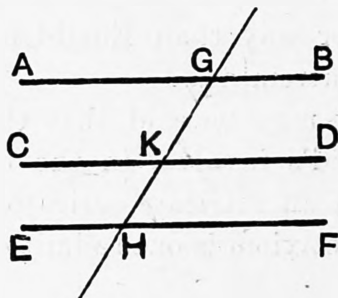
*Two intersecting straight lines cannot be both parallel to a third straight line.* The propositions in the *Elements* are then demonstrated as in Euclid up to I. 28, inclusive. Then, in I. 29, we proceed with Euclid up to the words, "therefore the angles BGH, GHD are together less than two right angles." We then infer that BGA and CHD must meet: because if a straight line be drawn through G so as to make the interior angles together equal to two right angles, this straight line will be parallel to CD, by I. 28; and, by our axiom, there cannot be two parallels to CD, both passing through G.

This form of making the necessary assumption has been recommended by various eminent mathematicians, among whom may be mentioned Playfair and De Morgan. It is hence known as **Playfair's Axiom**. By postponing the consideration of the axiom until it is wanted, that is, until after I. 28, and then presenting it in the form here given, the theory of parallel straight lines appears to be treated in the easiest manner that has hitherto been proposed.

PROPOSITION 30. THEOREM.

*Straight lines which are parallel to the same straight lines are parallel to each other.*

Let  $AB$ ,  $CD$  be each of them parallel to  $EF$  :  
then shall  $AB$  be parallel to  $CD$ .



**Construction.** Let the straight line  $GKH$  cut  $AB$ ,  $EF$ ,  $CD$ .

**Proof.** Because  $GKH$  cuts the parallel lines  $AB$ ,  $EF$ , the angle  $AGH =$  the alternate angle  $GHE$ . [I. 29.]

Again, because  $GK$  cuts the parallel lines  $EF$ ,  $CD$ , the exterior angle  $GKD =$  the interior opposite angle  $GHE$ . [I. 29.]

$\therefore$  the angle  $AGK =$  the angle  $GKD$ , [Axiom 1.]

and they are alternate angles ;

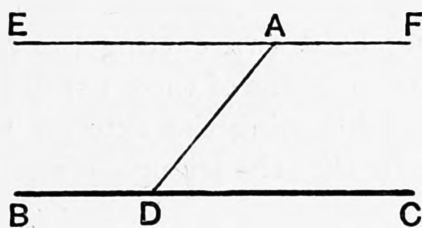
$\therefore AB$  is parallel to  $CD$ . [I. 27.]

Wherefore, *straight lines, etc.* [Q.E.D.]

## PROPOSITION 31. PROBLEM.

*To draw a straight line through a given point parallel to a given straight line.*

Let  $A$  be the given point, and  $BC$  the given straight line :  
*it is required to draw a straight line through the point  $A$  parallel to the straight line  $BC$ .*



**Construction.** In  $BC$  take any point  $D$ , and join  $AD$  ;  
 at the point  $A$  in the straight line  $AD$  make the angle  $DAE$   
 equal to the alternate angle  $ADC$ , [I. 23.]  
 and produce  $EA$  to  $F$ .  
 $EF$  shall be parallel to  $BC$ .

**Proof.** Because  $AD$ , which meets the two straight lines  $BC$ ,  
 $EF$ , makes the alternate angles  $EAD$ ,  $ADC$  equal. [Constr.]  
 $\therefore EF$  is parallel to  $BC$ . [I. 27.]

Wherefore *the straight line  $EAF$  is drawn through the given point  $A$  parallel to the given straight line  $BC$ .* [Q.E.F.]

## EXERCISES ON PROPOSITIONS 27–29.

1. In the figure of I. 16 prove that  $AB$  and  $FC$  are parallel.

\*\*2. Straight lines which are perpendicular to the same straight line are parallel.

**3.** Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

**4.** If two straight lines A and B are respectively parallel to two others C and D, shew that the inclination of A to B is equal to that of C to D.

**5.** A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is drawn and terminated by the parallel straight lines. Shew that the second straight line is bisected at the middle point of the first.

**\*\*6.** If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

**7.** If the straight line bisecting the exterior angle of a triangle be parallel to the base, shew that the triangle is isosceles.

### EXERCISES ON PROPOSITION 31.

**1.** Place between two parallel straight lines, and terminated by them, a straight line of given length.

**2.** Find a point B in a given straight line CD, such that if AB be drawn to B from a given point A, the angle ABC will be equal to a given angle.

[Draw AE parallel to CD and make the angle EAK equal to the given angle, AK being on the same side of AE as CD; AK meets CD in the required point B.]

**3.** Find a point such that the perpendiculars from it upon two given straight lines may be given. How many such points are there?

**4.** From a point D in the base BC of an isosceles triangle ABC a straight line DEF is drawn perpendicular to BC to meet the sides in E and F; prove that AEF is an isosceles triangle.

**5.** Through the middle point M of the base BC of a triangle a straight line DME is drawn, so as to cut off equal parts from the sides AB, AC, produced if necessary; show that BD is equal to CE.

[Through C draw CF parallel to AB to meet DE in F; prove that  $CF=BD$ , and also that  $CF=CE$ .]

**6.** Construct a triangle, having given the base, the altitude, and the length of the line joining the vertex to the middle point of the base.

**7.** Find points D, E in the sides AB, AC of a triangle ABC, such that DE is parallel to BC and equal to BD.

[BE bisects the angle ABC.]

**8.** Given the altitude and the base angles of a triangle, construct it.

## PROPOSITION 32. THEOREM.

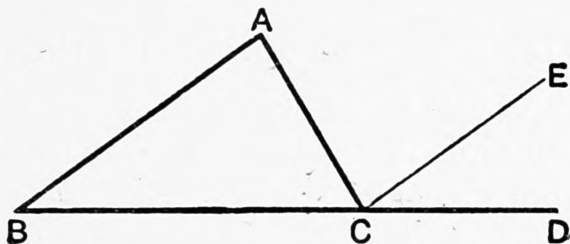
*If a side of any triangle be produced,*

*(1) the exterior angle is equal to the two interior and opposite angles ;  
and (2) the three interior angles of the triangle are together equal to two right angles.*

Let  $ABC$  be a triangle, and let one of its sides  $BC$  be produced to  $D$  :

*then shall (1) the exterior angle  $ACD$  be equal to the two interior and opposite angles  $CAB, ABC$  ;*

*and (2) the three interior angles  $ABC, BCA, CAB$  shall be together equal to two right angles.*



**Construction.** Through the point  $C$  draw  $CE$  parallel to  $AB$ . [I. 31.]

**Proof.** (1) Because  $AB$  is parallel to  $CE$ , and  $AC$  meets them, the alternate angles  $BAC, ACE$  are equal. [I. 29.]

Again, because  $AB$  is parallel to  $CE$ , and  $BD$  meets them, the exterior angle  $ECD$  is equal to the interior and opposite angle  $ABC$ . [I. 29.]

$\therefore$  the whole exterior angle  $ACD =$  the two interior and opposite angles  $CAB, ABC$ . [Axiom 2.]

(2) To each of these equals add the angle  $ACB$  ;  
 $\therefore$  the angles  $ACD, ACB =$  the three angles  $CBA, BAC, ACB$ . [Axiom 2.]

But the angles  $ACD, ACB$  together  $=$  two right angles ; [I. 13.]

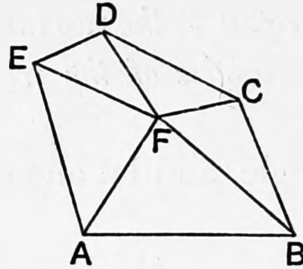
$\therefore$  also the angles  $CBA, BAC, ACB$  together  $=$  two right angles. [Axiom 1.]

Wherefore, *if a side of any triangle, etc.*

[Q.E.D.]



**Corollary 1.** *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Take any rectilineal figure, and join any point  $F$  within it to its angular points.

We thus have as many triangles as there are sides to the figure.

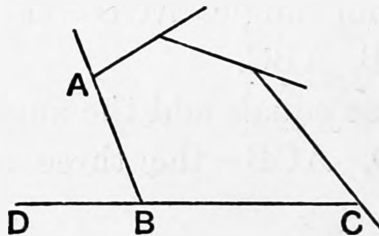
Also, by the preceding propositions, the three angles of each triangle make up two right angles.

$\therefore$  all the angles of these triangles = twice as many right angles as the figure has sides.

But all the angles of the triangles = the interior angles of the figure, together with the angles at the point  $F$ , which are equal to four right angles. [I. 15. Cor. 2.]

$\therefore$  all the interior angles of the figure, together with four right angles, = twice as many right angles as the figure has sides.

**Corollary 2.** *All the exterior angles of any rectilineal figure are together equal to four right angles.*



Because every interior angle  $ABC$ , with its adjacent exterior angle  $ABD$ , is equal to two right angles ; [I. 13.]

$\therefore$  all the interior angles of the figure, together with all its

exterior angles, = twice as many right angles as the figure has sides.

But all the interior angles of the figure, together with four right angles, = twice as many right angles as the figure has sides. [Corollary 1.

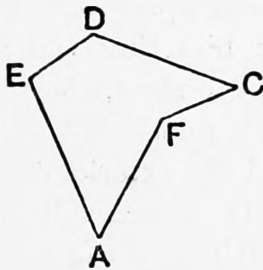
$\therefore$  all the interior angles of the figure, together with all its exterior angles, = all the interior angles of the figure, together with four right angles.

$\therefore$  all the exterior angles = four right angles.

### NOTE ON THE COROLLARIES TO I. 32.

I. 32. The corollaries to I. 32 were added by Simson. In the second corollary it ought to be stated what is meant by an *exterior* angle of a rectilinear figure. At each angular point let *one* of the sides meeting at that point be produced; then the exterior angle at that point is the angle contained between this produced part and the side which is not produced. *Either* of the sides may be produced, for the two angles which can thus be obtained are equal, by I. 15.

The rectilinear figures to which Euclid confines himself are those in which the angles all face inwards; we may here however notice another class of figures. In the accompanying diagram the angle AFC faces



outwards, and it is an angle less than two right angles; this angle however is not one of the interior angles of the figure AEDCF. We may consider the corresponding interior angle to be the excess of four right angles above the angle AFC.

An angle such as AFC, greater than two right angles, is called a **re-entrant angle**.

The *first* of the corollaries to I. 32 is true for a figure which has a re-entrant angle or re-entrant angles; but the *second* is not.

## EXERCISES.

**\*\*1.** If two triangles have two angles of the one equal to two angles of the other, the third angle of the one is equal to the third angle of the other.

**\*\*2.** Each angle of an equilateral triangle is equal to two-thirds of a right angle. Hence trisect a right angle.

**\*\*3.** The sum of the angles of any quadrilateral figure is equal to four right angles.

**\*\*4.** If one angle of a triangle be equal to the sum of the other two, the triangle is right-angled; if it be less than the sum of the other two it is acute; if greater, then obtuse.

**5.** In an obtuse-angled triangle the side opposite the obtuse angle is the greatest side.

**6.** What is the magnitude of an angle of a regular (1) pentagon, (2) hexagon, and (3) octagon?

**7.** The bisectors of the exterior angles of a quadrilateral form a quadrilateral, the sum of whose two opposite angles is two right angles.

**\*\*8.** The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

[A being the right  $\angle$  of the  $\triangle ABC$ , make  $\angle BAD = \angle ABC$ , and let AD meet BC in D; then  $\angle DAC = \angle DCA$ , etc.]

**9.** From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; shew that the angles made by them with the base are each equal to half the vertical angle.

**10.** If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

**11.** ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD, CD respectively, meeting at D; shew that the angle BDC together with half the angle BAC make up a right angle.

**12.** The angle between the internal bisector of one base angle and the external bisector of the other is equal to one-half the vertical angle.

**13.** The angle included between the bisector of the angle A of a triangle and the perpendicular from A upon the opposite side is half the difference of the base angles of the triangle.

**14.** The bisector of the exterior vertical angle of a triangle makes with the base an angle equal to half the difference of the base angles, and with either side an angle equal to half the sum of the base angles.

**15.** On the sides of any triangle  $ABC$  equilateral triangles  $BCD$ ,  $CAE$ ,  $ABF$  are described, all external; shew that the straight lines  $AD$ ,  $BE$ ,  $CF$  are all equal.

**16.** Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.

[Through the points draw straight lines making with the given line angles equal to two-thirds of a right angle.]

**17.**  $A$  is the vertex of an isosceles triangle  $ABC$ , and  $BA$  is produced to  $D$ , so that  $AD$  is equal to  $BA$ , and  $DC$  is drawn; shew that  $BCD$  is a right angle.

**18.** The median passing through the vertex of a triangle is equal to, greater than, or less than half of the base according as the vertical angle is a right, an acute, or an obtuse angle. [Use Ex. 4.]

**19.** If one angle of a triangle be triple another the triangle may be divided into two isosceles triangles.

[Let  $ABC$  be the  $\triangle$  where  $\angle BCA = 3\angle ABC$ ; at  $C$  make  $\angle BCD = \angle ABC$ , and let  $CD$  meet  $AB$  in  $D$ . Then  $DBC$ ,  $ADC$  can be proved to be isosceles triangles.]

**20.** If one angle of a triangle be double another, an isosceles triangle may be added to it so as to form together with it a single isosceles triangle.

[Let  $ABC$  be the  $\triangle$  where  $\angle BCA = 2\angle ABC$ ; with centre  $A$  and radius  $AB$  describe a circle meeting  $BC$  produced in  $D$ ; then  $ACD$  will be the  $\triangle$  to be added.]

**21.** Given two angles of a triangle and the side opposite one of them, construct the triangle.

**22.** Within a triangle  $ABC$  straight lines  $AD$ ,  $BE$ , and  $CF$  are drawn, making with  $AB$ ,  $BC$ , and  $CA$  respectively the angles  $DAB$ ,  $EBC$ ,  $FCA$  equal to each other. If  $AD$ ,  $BE$ , and  $CF$  do not meet in one point they will form by their intersections a triangle whose angles are equal to those of the triangle  $ABC$ .

**23.** Through the vertex  $A$  of a triangle  $ABC$  a straight line is drawn parallel to  $BC$ . If, on either side of  $A$ , lengths  $AD$ ,  $AE$  be measured off from it, each equal to  $AC$ , and  $CD$ ,  $CE$  be joined, prove that  $DCE$  is a right angle.

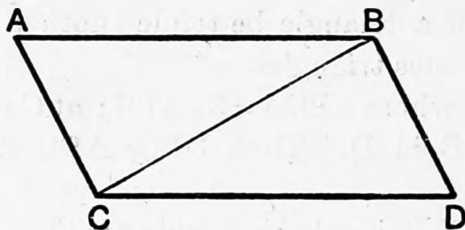
**\*\*24.** Each of the angles of a polygon of  $n$  sides, whose angles are all equal, is equal to  $\frac{2n-4}{n}$  right angles.

## PROPOSITION 33. THEOREM.

*The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.*

Let AB and CD be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AC and BD :

*then shall AC and BD be equal and parallel.*



**Construction.** Join BC.

**Proof.** Because AB is parallel to CD, [Hypothesis.  
and BC meets them,  
the alternate angles ABC, BCD are equal. [I. 29.

Then, in the triangles ABC, BCD,

because  $\begin{cases} AB = CD, \\ \text{and } BC \text{ is common,} \\ \text{and the angle } ABC = \text{the angle } BCD; \end{cases}$  [Hypothesis.  
[Proued.

$\therefore$  the triangles are equal in all respects, so that

the base AC = the base BD,

and the angle ACB = the angle CBD. [I. 4.

Also these are alternate angles ;

$\therefore$  AC is parallel to BD, [I. 27.

and it was shewn to be equal to it.

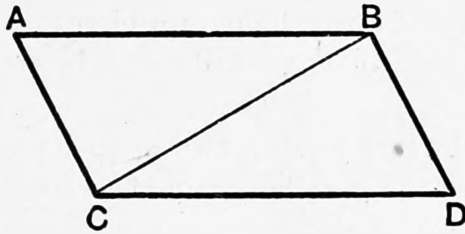
Wherefore, *the straight lines, etc.*

[Q. E. D.

## PROPOSITION 34. THEOREM.

*The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram, that is, divides it into two equal parts.*

Let ACDB be a parallelogram, of which BC is a diagonal : then (1) the opposite sides and angles of the figure shall be equal to one another, and (2) the diagonal BC shall bisect it.



**Proof.** (1) Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal. [I. 29.  
And because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal. [I. 29.

Then, in the triangles ABC, DCB,

because  $\left\{ \begin{array}{l} \text{the angle } ABC = \text{angle } DCB, \\ \text{and the angle } BCA = \text{angle } CBD, \\ \text{and the side } BC \text{ adjacent to the equal angles in each} \\ \text{is common to both ;} \end{array} \right.$

$\therefore$  the triangles are equal in all respects, so that

$\left\{ \begin{array}{l} AB = CD, \\ AC = BD, \\ \text{and the angle } BAC = \text{the angle } CDB. \end{array} \right.$  [I. 26.

And because the angle ABC = the angle BCD,

and the angle CBD = the angle ACB,

$\therefore$  the whole angle ABD = the whole angle ACD. [Ax. 2.

And the angle BAC has been proved equal to the angle CDB.

$\therefore$  the opposite sides and angles are equal.

(2) Also it has been proved that the triangles ABC, DCB are equal in all respects ;

$\therefore$  the diagonal BC bisects the parallelogram ACDB.

Wherefore, the opposite sides, etc.

[Q. E. D.

## EXERCISES.

**1.** If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

**\*\*2.** If the opposite angles of a quadrilateral are equal it is a parallelogram. [ $A + B = C + D = \text{two rt. } \angle^s$ , by I. 32, Cor. 2;  $\therefore$  etc.]

**\*\*3.** If the opposite sides of a quadrilateral are equal it is a parallelogram.

**\*\*4.** The diagonals of a parallelogram bisect each other.

**\*\*5.** If the diagonals of a quadrilateral bisect each other it is a parallelogram.

**\*\*6.** If the straight line joining two opposite angles of a parallelogram bisect the angles the parallelogram is a rhombus.

**7.** Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length. [Use Ex. 1, page 56.]

**\*\*8.** Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

**9.** Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

**\*\*10.** If the diagonals of a parallelogram are equal all its angles are equal, and it is a rectangle.

**11.** Shew that any straight line passing through the middle point of the diagonal of a parallelogram and terminated by two opposite sides, bisects the parallelogram.

**12.** Bisect a parallelogram by a straight line drawn through any given point. [Use Ex. 11.]

**\*\*13.** The diagonals of a rhombus are at right angles.

**14.** A, B, C are three points in a straight line, such that AB is equal to BC; shew that the sum of the perpendiculars from A and C on any straight line which does not pass between A and C is double the perpendicular from B on the same straight line.

**15.** If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles. [Use Ex. 14.]

**\*\*16.** The parallel to any side of a triangle through the middle point of another bisects the third side. [See Appendix, Art. 1.]

**17.** On the sides of a parallelogram ABCD four points E, F, G and H are taken, and parallels to the sides are drawn through these points



to form a parallelogram PQRS. Prove that the sum of the areas ABCD and PQRS is equal to twice that of EFGH.

[Twice EFGH = twice PQRS +  $\parallel^{\text{gms}}$  PB, QC, RD, SA = etc.]

**18.** In the base BC of an isosceles triangle ABC any point D is taken, and perpendiculars are drawn from D upon the two sides AB, AC. Prove that the sum of these two perpendiculars is equal to the perpendicular from B upon AC.

[Let DE, DF be the perpendiculars, and draw BK perp<sup>r</sup> to AC and DL perp<sup>r</sup> to BK; prove the two  $\triangle^s$  BED, DLB equal in all respects and  $\therefore$  BL = DE, etc.]

**19.** From a given point O draw to a given straight line AB a straight line that shall be bisected by another given straight line AC.

[Draw OP  $\parallel^1$  to AB to meet AC in P, and PQ  $\parallel^1$  to OA to meet AB in Q; then OQ is the required straight line, by Ex. 4.]

**20.** It is required to draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines.

**\*\*21.** AB, AC are two given straight lines; through a given point E between them it is required to draw a straight line GEH such that the intercepted portion GH shall be bisected at the point E. [See Appendix, Art. 18.]

*Note.*—The Propositions from I. 35 to I. 48 may be said to constitute the third section of the first Book of the *Elements*. They relate to equality of area in figures which are not necessarily identical in form.

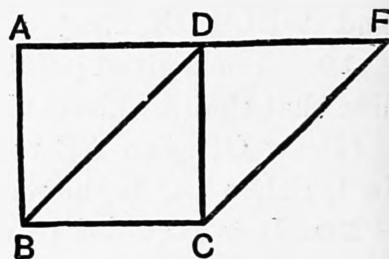


## PROPOSITION 35. THEOREM.

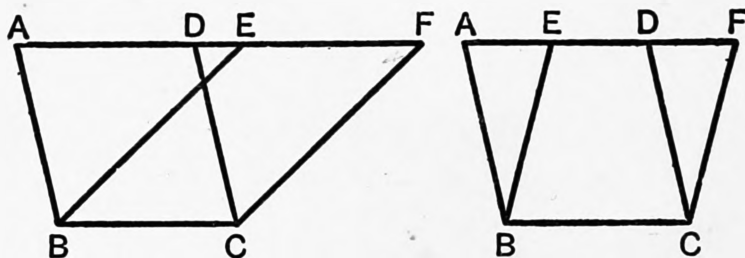
*Parallelograms on the same base, and between the same parallels, are equal to one another.*

Let the parallelograms  $ABCD$ ,  $EBCF$  be on the same base  $BC$ , and between the same parallels  $AF$ ,  $BC$  :  
*then shall the parallelogram  $ABCD$  be equal to the parallelogram  $EBCF$ .*

**Proof.** CASE I. If the sides  $AD$ ,  $DF$  opposite to the base  $BC$  be terminated at the same point  $D$ , it is plain that each of the parallelograms is double of the triangle  $BDC$ ; [I. 34. and they are therefore equal. [Ax. 6.



CASE II. But if the sides  $AD$ ,  $EF$ , opposite to the base  $BC$  be not terminated at the same point, then because  $ABCD$  is a parallelogram,  $AD = BC$ ; [I. 34.



for the same reason  $EF = BC$ ;  $\therefore AD = EF$ ; [Axiom 1.  
 $\therefore$  the whole, or the remainder,  $AE =$  the whole, or the remainder,  $DF$ . [Axioms 2, 3.

Then, in the triangles  $EAB$ ,  $FDC$ ,

because  $\begin{cases} AB = DC, \\ \text{and } AE = DF, \\ \text{and the exterior angle } FDC = \text{the interior opposite} \\ \text{angle } EAB; \end{cases}$  [I. 29.  
 $\therefore$  the  $\triangle EAB =$  the  $\triangle FDC$ . [I. 4.

Take the  $\triangle FDC$  from the figure  $ABCF$ , and from the same, or a similar, figure take the  $\triangle EAB$ , and the remainders are equal; [Axiom 3.

that is, the parallelogram  $ABCD =$  the parallelogram  $EBCF$ .

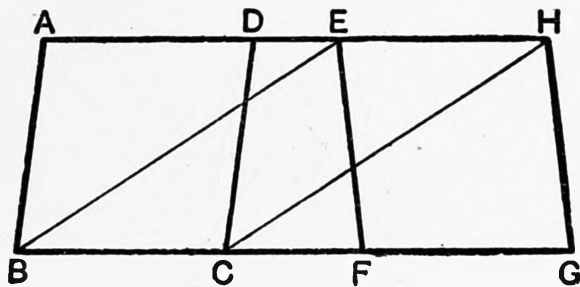
Wherefore, *parallelograms on the same base, etc.*

[Q.E.D.

## PROPOSITION 36. THEOREM.

*Parallelograms on equal bases, and between the same parallels, are equal to one another,*

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG :  
*then shall ABCD be equal to EFGH.*



**Construction.** Join BE, CH.

**Proof.** Because  $BC = FG$ , [Hypothesis.  
 and  $FG = EH$ , [I. 34.  
 $\therefore BC = EH$ , [Axiom 1.  
 and they are parallels ; [Hypothesis.

$\therefore$  BE and CH, which join their extremities toward the same parts, are both equal and parallel. [I. 33.

$\therefore$  EBCH is a parallelogram, [Definition.  
 and it is equal to ABCD, because they are on the same base BC, and between the same parallels BC, AH. [I. 35.

Also the parallelogram EFGH is equal to the same EBCH, since they are on the same base EH, and between the same parallels EH, BG. [I. 35.

$\therefore$  the parallelogram ABCD = the parallelogram EFGH. [Ax. 1.

Wherefore, *parallelograms, etc.* [Q.E.D.

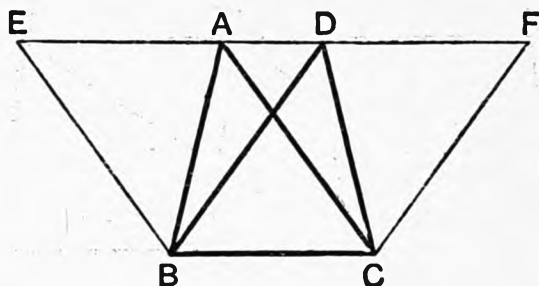
## EXERCISES.

1. Construct a rectangle equal to a given parallelogram.
2. Construct a rhombus equal to a given parallelogram.
3. Divide a parallelogram into four equal parallelograms.
4. If two adjacent sides of a parallelogram be given, its area is greatest when the sides are perpendicular.

## PROPOSITION 37. THEOREM.

*Triangles on the same base, and between the same parallels, are equal.*

Let the triangles  $ABC$ ,  $DBC$  be on the same base  $BC$ , and between the same parallels  $AD$ ,  $BC$  :  
*then shall the triangle  $ABC$  be equal to the triangle  $DBC$ .*



**Construction.** Produce  $AD$  both ways to the points  $E$ ,  $F$  ;  
 through  $B$  draw  $BE$  parallel to  $CA$ , and through  $C$  draw  
 $CF$  parallel to  $BD$ . [I. 31.]

**Proof.** The parallelograms  $EBCA$  and  $DBCF$  are equal,  
 because they are on the same base  $BC$ , and between the same  
 parallels  $BC$ ,  $EF$ . [I. 35.]

Also the  $\triangle ABC$  is half of the parallelogram  $EBCA$ , because  
 the diagonal  $AB$  bisects it, [I. 34.]

and the  $\triangle DBC$  is half of the parallelogram  $DBCF$ , because  
 the diagonal  $DC$  bisects it ; [I. 34.]

$\therefore$  the  $\triangle ABC =$  the  $\triangle DBC$ . [Axiom 7.]

Wherefore, *triangles, etc.* [Q.E.D.]

## EXERCISES.

1.  $PQR$  is a straight line parallel and equal to the base  $BC$  of a triangle  $ABC$  and meets the sides in  $P$  and  $Q$ . Prove that the triangles  $BPQ$ ,  $AQR$  are equal.

[Draw  $QS$  parallel to  $AB$  to meet  $BC$  in  $S$  ; then  $QR = SC$  ;  $\therefore RC$  is parallel to  $QS$  or  $AB$  ;  $\therefore \triangle^s AQR, RQC = \triangle ARC = \frac{1}{2}BR = \frac{1}{2}BQ + \frac{1}{2}SR = \triangle^s BPQ, RQC$ , etc.]

2. Describe a triangle equal to a given quadrilateral  $ABCD$ .

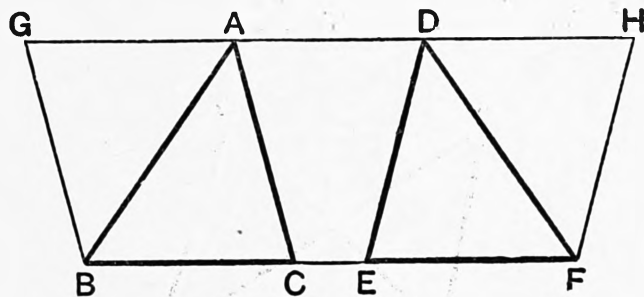
[Draw  $AE$  parallel to  $DB$  to meet  $BC$  produced in  $E$  ; then  $DEC$  is the  $\triangle$  required.]

## PROPOSITION 38. THEOREM.

*Triangles on equal bases, and between the same parallels, are equal to one another.*

Let the triangles  $ABC$ ,  $DEF$  be on equal bases  $BC$ ,  $EF$ , and between the same parallels  $BF$ ,  $AD$ :

*then shall the triangle  $ABC$  be equal to the triangle  $DEF$ .*



**Construction.** Produce  $AD$  both ways to the points  $G$ ,  $H$ ; through  $B$  draw  $BG$  parallel to  $CA$ , and through  $F$  draw  $FH$  parallel to  $ED$ . [I. 31.]

**Proof.** Each of the figures  $GBCA$ ,  $DEFH$  is a parallelogram, [Definition.]

and they are equal because they are on equal bases  $BC$ ,  $EF$ , and between the same parallels  $BF$ ,  $GH$ . [I. 36.]

Also the  $\triangle ABC$  is half of the parallelogram  $GBCA$ , because the diagonal  $AB$  bisects it; [I. 34.]

and the  $\triangle DEF$  is half of the parallelogram  $DEFH$ , because the diagonal  $DF$  bisects it;

$\therefore$  the  $\triangle ABC =$  the  $\triangle DEF$ .

[Axiom 7.]

Wherefore, *triangles, etc.*

[Q.E.D.]

## EXERCISES.

**\*\*1.** A triangle is bisected by either of its medians.

**2.**  $ABC$  is a triangle and  $E$  any point in the median  $AD$ ; prove that the triangles  $ABE$ ,  $ACE$  are equal.

**\*\*3.** If two triangles have two sides of the one equal to two sides of the other, and the contained angles supplementary, the triangles are equal in area.

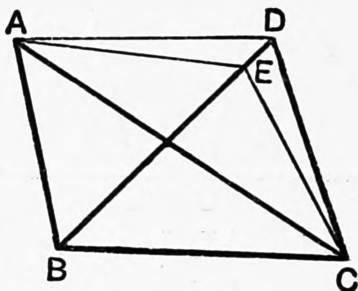
[For they can be placed so that they have one equal side of each coincident, and the other two equal sides in the same straight line. Then apply Ex. 1.]

**\*\*4.** Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area. [Use Ex. 4, page 64.]

## PROPOSITION 39. THEOREM.

*Equal triangles on the same base, and on the same side of it, are between the same parallels.*

Let the equal triangles  $ABC$ ,  $DBC$  be on the same base  $BC$ , and on the same side of it:  
*they shall be between the same parallels.*



**Construction.** Join  $AD$ .

$AD$  shall be parallel to  $BC$ .

For if it is not, let  $AE$  be parallel to  $BC$ , and let it meet  $BD$  at  $E$ ,

[I. 31.]

and join  $EC$ .

**Proof.** The  $\triangle ABC = \text{the } \triangle EBC$ , because they are on the same base  $BC$ , and between the same parallels  $BC$ ,  $AE$ .

[I. 37.]

But the  $\triangle ABC = \text{the } \triangle DBC$ ;

[Hypothesis.]

$\therefore$  also the  $\triangle DBC = \text{the } \triangle EBC$ ,

[Axiom 1.]

the greater to the less, which is impossible;

$\therefore$   $AE$  is not parallel to  $BC$ .

In the same manner it can be shewn that no other straight line through  $A$  except  $AD$  is parallel to  $BC$ ;

$\therefore$   $AD$  is parallel to  $BC$ .

Wherefore, *equal triangles, etc.*

[Q.E.D.]

## PROPOSITION 40. THEOREM.

*Equal triangles, on equal bases, in the same straight line, and on the same side of it, are between the same parallels.*

Let the equal triangles  $ABC$ ,  $DEF$  be on equal bases  $BC$ ,  $EF$ , in the same straight line  $BF$ , and on the same side of it: *they shall be between the same parallels.*

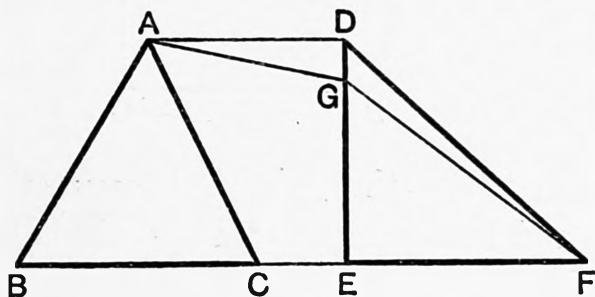
**Construction.** Join  $AD$ .

$AD$  shall be parallel to  $BF$ .

For if it is not, let  $AG$  be drawn parallel to  $BF$ , and let it meet  $ED$  at  $G$ ,

[I. 31.]

and join  $GF$ .



**Proof.** The  $\triangle ABC = \text{the } \triangle GEF$ , because they are on equal bases  $BC$ ,  $EF$ , and between the same parallels. [I. 38.]

But the  $\triangle ABC = \text{the } \triangle DEF$ ;

[Hypothesis.]

$\therefore$  also the  $\triangle DEF = \text{the } \triangle GEF$ ,

[Axiom 1.]

the greater to the less, which is impossible.

$\therefore$   $AG$  is not parallel to  $BF$ .

In the same manner it can be shewn that no other straight line through  $A$  except  $AD$  is parallel to  $BF$ ;

$\therefore$   $AD$  is parallel to  $BF$ .

Wherefore, *equal triangles, etc.*

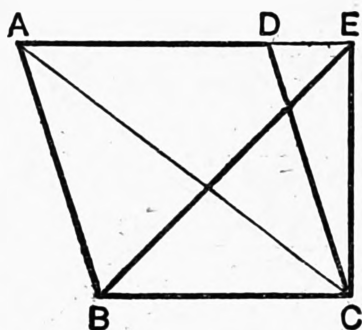
[Q.E.D.]

## PROPOSITION 41. THEOREM.

*If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.*

Let the parallelogram ABCD and the triangle EBC be on the same base BC, and between the same parallels BC, AE: the parallelogram ABCD shall be double of the triangle EBC.

**Construction.** Join AC.



**Proof.** The  $\triangle ABC =$  the  $\triangle EBC$ , because they are on the same base BC, and between the same parallels BC, AE. [I. 37. But the parallelogram ABCD is double of the  $\triangle ABC$ , because the diagonal AC bisects the parallelogram. [I. 34.  $\therefore$  the parallelogram ABCD is also double of the  $\triangle EBC$ .

Wherefore, *if a parallelogram, etc.*

[Q. E. D.]

## EXERCISES ON PROPOSITION 39.

**\*\*1.** Two straight lines AB and CD intersect at E, and the triangle AEC is equal to the triangle BED: shew that BC is parallel to AD.

**\*\*2.** The straight line which joins the middle points of two sides of any triangle is parallel to the base and is equal to half the base. [See Appendix, Art. 1.]

**\*\*3.** Straight lines joining the middle points of adjacent sides of a quadrilateral form a parallelogram. [Use Ex. 2.]

**4.** In the base AC of a triangle take any point D; bisect AD, DC, AB, BC at the points E, F, G, H respectively: shew that EG is equal and parallel to FH. [Use Ex. 2.]

5. Two triangles of equal area stand on the same base and on opposite sides : shew that the straight line joining their vertices is bisected by the base or the base produced. [See App., Art. 2.]

6. If a quadrilateral figure be bisected by one diagonal the second diagonal is bisected by the first. [Use Ex. 5.]

7. Any quadrilateral figure which is bisected by both of its diagonals is a parallelogram.

### EXERCISES ON PROPOSITION 40.

R 1. A quadrilateral is divided into four triangles by its diagonals ; prove that if two adjacent triangles be equal the other two triangles will also be equal.

2. The straight lines AD, BE bisecting the sides BC, AC of a triangle intersect at G : shew that AG is double of GD. [See App., Art. 7.]

### EXERCISES ON PROPOSITION 41.

1. ABCD is a quadrilateral having BC parallel to AD ; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of DC a straight line parallel to AB to meet AD and BC.

2. ABCD is a quadrilateral having BC parallel to AD, E is the middle point of DC ; shew that the triangle AEB is half the quadrilateral.

3. If any point be taken within a parallelogram the sum of the triangles formed by joining the point with the extremities of a pair of opposite sides is half the parallelogram.

[Through the point draw a straight line parallel to the two opposite sides to meet the other sides.]

4. ABCD is a parallelogram ; from any point P in the diagonal BD the straight lines PA, PC are drawn. Shew that the triangles PAB and PCB are equal in area.

[By Ex. 3,  $\triangle BPC + \triangle APD = \frac{1}{2}ABCD = \triangle ABD = \triangle APB + \triangle APD$ .]

\*\*5. If the middle points of any two sides of a triangle be joined, the triangle so cut off is one quarter of the whole.

6. On the same side of the straight line ABC equal rectangles ABDE, ACFG are described. Prove that BG and DF are parallel.

[ $\triangle GBF = \frac{1}{2} \text{ rect. AF} = \frac{1}{2} \text{ rect. AD} = \triangle GDB$  ;  $\therefore$  etc.]

\*\*7. If two sides of a triangle be given the area of the triangle is greatest when the angle between them is a right angle.

8. Through the vertices of a quadrilateral straight lines are drawn parallel to the diagonals ; prove that the figure thus obtained is a parallelogram whose area is twice that of the quadrilateral.

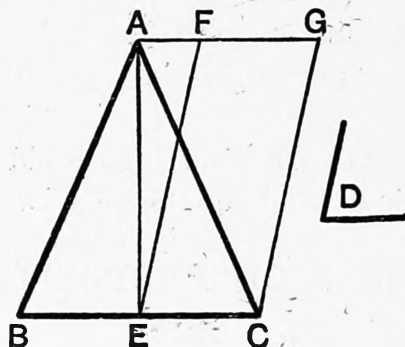


## PROPOSITION 42. PROBLEM.

*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

Let  $ABC$  be the given triangle, and  $D$  the given rectilineal angle :

*it is required to describe a parallelogram that shall be equal to the given triangle  $ABC$ , and have one of its angles equal to  $D$ .*



**Construction.** Bisect  $BC$  at  $E$ ; [I. 10.  
join  $AE$ , and at the point  $E$ , in the straight line  $EC$ , make  
the angle  $CEF$  equal to  $D$ ; [I. 23.  
through  $A$  draw  $AFG$  parallel to  $EC$ , and through  $C$  draw  
 $CG$  parallel to  $EF$ . [I. 31.  
Then  $FECG$  is the parallelogram constructed as required.

**Proof.** The  $\triangle ABE =$  the  $\triangle AEC$ , because they are on equal  
bases  $BE, EC$ , and between the same parallels  $BC, AG$ . [I. 38.  
 $\therefore$  the  $\triangle ABC$  is double of the  $\triangle AEC$ .

But the parallelogram  $FECG$  is also double of the  $\triangle AEC$ ,  
because they are on the same base  $EC$ , and between the same  
parallels  $EC, AG$ . [I. 41.

$\therefore$  the parallelogram  $FECG =$  the  $\triangle ABC$ , [Axiom 6.  
and its angle  $CEF$  is equal to the given angle  $D$ . [Construction.

Wherefore a parallelogram  $FECG$  has been described equal to  
the given triangle  $ABC$ , and having one of its angles  $CEF$  equal to  
the given angle  $D$ . [Q. E. F.

## EXERCISE.

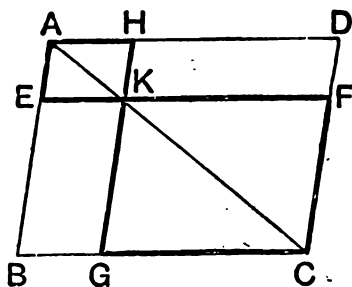
Construct a triangle equal to a given parallelogram, and having an  
angle equal to a given angle.

## PROPOSITION 43. THEOREM.

*The complements of the parallelograms which are about the diagonal of any parallelogram are equal.*

Let ABCD be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms which make up the whole figure ABCD, and which are therefore called the complements:

*the complement BK shall be equal to the complement KD.*



**Proof.** Because AEKH is a parallelogram, and AK its diagonal, the  $\triangle AEK = \triangle AHK$ . [I. 34.]

For the same reason the  $\triangle KGC = \triangle KFC$ ;  
 $\therefore$  the two triangles AEK, KGC together = the two triangles AHK, KFC. [Axiom 2.]

But the whole  $\triangle ABC = \triangle ADC$ , because the diagonal AC bisects the parallelogram ABCD; [I. 34.]

$\therefore$  the remainder, the complement BK, is equal to the remainder, the complement KD. [Axiom 3.]

Wherefore, *the complements, etc.* [Q. E. D.]

## EXERCISES.

1. Each of the parallelograms about the diagonal of a rhombus is a rhombus.

In the figure of I. 43 prove that

2. EH, DB, GF are parallel.

[ $\triangle BED = \triangle BCE = \frac{1}{2}BF = \frac{1}{2}CH = \triangle HCD = \triangle BHD$ , etc.]

3. The triangle EKC is half of the complement BK.

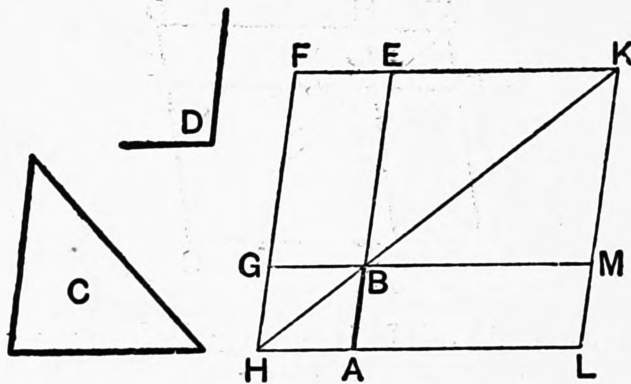
4. The triangles EKC, HKC are equal.

## PROPOSITION 44. PROBLEM.

*To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

Let AB be the given straight line, C the given triangle, and D the given rectilineal angle :

*it is required to apply to AB a parallelogram equal to the triangle C, and having an angle equal to D.*



**Construction.** Make the parallelogram BEFG equal to the  $\triangle C$ , and having the angle EBG equal to the angle D, so that BE may be in the same straight line with AB ; [I. 42.]

produce FG to H ;

through A draw AH parallel to BG or EF, [I. 31.]  
and join HB.

Because the straight line HF meets the parallels AH, EF, the angles AHF, HFE are together equal to two right angles ; [I. 29.]

Therefore the angles BHF, HFE are together less than two right angles ;

$\therefore$  HB and FE will meet towards B and E, if produced far enough ; [Axiom 12.]

let them meet at K.

Through K draw KL parallel to EA or FH ; [I. 31.]

and produce HA, GB to meet KL in the points L, M.

**Proof.** HLKF is a parallelogram, of which the diagonal is HK ; and AG, ME are parallelograms about HK ; and LB, BF are the complements ;

$$\therefore LB = BF ; \quad [I. 43.]$$

$$\text{But } BF = \text{the } \triangle C ; \quad [\text{Construction.}]$$

$$\therefore LB = \text{the } \triangle C. \quad [\text{Axiom 1.}]$$

$$\text{And because the angle } GBE = \text{the angle } ABM, \quad [I. 15.]$$

$$\text{and likewise} = \text{the angle } D ; \quad [\text{Construction.}]$$

$$\therefore \text{the angle } ABM = \text{the angle } D. \quad [\text{Axiom 1.}]$$

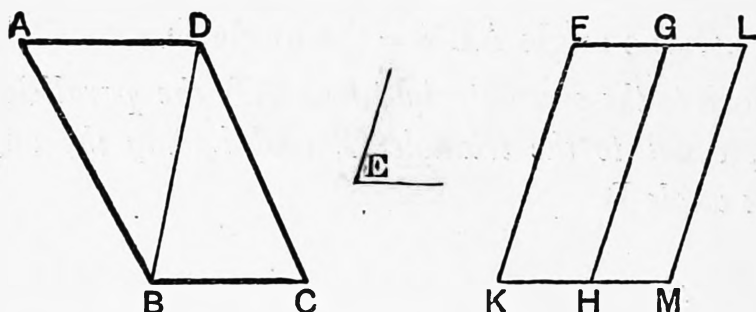
Wherefore *to the given straight line AB the parallelogram LB is applied, equal to the triangle C, and having the angle ABM equal to the angle D.* [Q.E.F.]

## PROPOSITION 45. PROBLEM.

*To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.*

Let  $ABCD$  be the given rectilineal figure, and  $E$  the given rectilineal angle :

*it is required to describe a parallelogram equal to  $ABCD$ , and having an angle equal to  $E$ .*



**Construction.** Join  $DB$ , and describe the parallelogram  $FH$  equal to the  $\triangle ADB$ , and having the angle  $FKH$  equal to the angle  $E$  ; [I. 42.

and to the straight line  $GH$  apply the parallelogram  $GM$  equal to the  $\triangle DBC$ , and having the angle  $GHM$  equal to the angle  $E$ . [I. 44.

The figure  $FKML$  shall be the parallelogram required.

**Proof.** Because the angle  $E$  = each of the angles  $FKH$ ,  $GHM$ ,

the angle  $FKH$  is equal to the angle  $GHM$ . [Axiom 1.

Add to each of these equals the angle  $KHG$  ;

$\therefore$  the angles  $FKH$ ,  $KHG$  are equal to the angles  $KHG$ ,  $GHM$ , [Axiom 2.

But  $FKH$ ,  $KHG$  together = two right angles ; [I. 29.

$\therefore$   $KHG$ ,  $GHM$  together = two right angles.

$\therefore$   $KH$  is the same straight line with  $HM$ . [I. 14.

And because  $HG$  meets the parallels  $KM$ ,  $FG$ , the alternate angles  $MHG$ ,  $HGF$  are equal. [I. 29.

Add to each of these the angle  $HGL$  ;

$\therefore$  the angles MHG, HGL together = the angles HGF, HGL.

[Axiom 2.

But MHG, HGL together = two right angles ; [I. 29.

$\therefore$  HGF, HGL together = two right angles ;

$\therefore$  FG, GL are in the same straight line. [I. 14.

And because KF is parallel to HG, and HG to ML, [Constr.

$\therefore$  KF is parallel to ML, [I. 30.

and KM, FL are parallels ; [Construction.

$\therefore$  KFLM is a parallelogram. [Definition.

And because the  $\triangle ABD$  = the parallelogram HF, [Constr.

and the  $\triangle DBC$  = the parallelogram GM, [Constr.

$\therefore$  the whole rectilineal figure ABCD = the whole parallelogram KFLM. [Axiom 2.

Wherefore *the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E.* [Q.E.F.

**Corollary.** From this it is clear how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle ; and so on. [I. 44.

### EXERCISE.

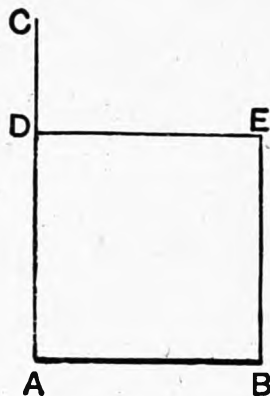
Describe a triangle equal to a given rectilineal figure.

[Let ABCDE be the given figure ; join BD, AD and draw CP, EQ parallel to DB, DA respectively to meet AB produced if necessary in P and Q ; then  $\triangle DCB = \triangle DPB$  and  $\triangle DEA = \triangle DQA$  [I. 37], and  $\therefore$  whole figure =  $\triangle DQP$ . Similarly for a figure with a larger number of sides.]

## PROPOSITION 46. PROBLEM.

*To describe a square on a given straight line.*

Let  $AB$  be the given straight line :  
*it is required to describe a square on  $AB$ .*



**Construction.** From the point  $A$  draw  $AC$  at right angles to  $AB$ , [I. 11.]  
 and make  $AD$  equal to  $AB$ ; [I. 3.]  
 through  $D$  draw  $DE$  parallel to  $AB$ , and through  $B$  draw  $BE$  parallel to  $AD$ . [I. 31.]  
 $ADEB$  shall be a square.

**Proof.**  $ADEB$  is by construction a parallelogram ;

$AB = DE$ , and  $AD = BE$ . [I. 34.]

But  $AB = AD$ . [Construction.]

$\therefore BA, AD, DE, EB$  are all equal, and the parallelogram  $ADEB$  is therefore equilateral. [Axiom 1.]

Also the angle  $BAD$  is a right angle; [Construction.]

$\therefore$  the figure  $ADEB$  is equilateral, and it has one angle a right angle.

Therefore it is a square,

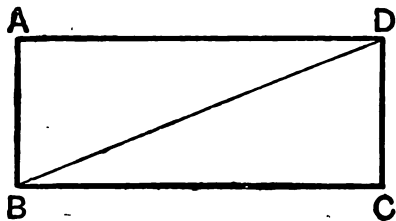
[Definition 30.]

and it is described on the given straight line  $AB$ .

[Q. E. F.]

## NOTE TO PROPOSITION 46.

It can be proved that squares and also rectangles have all their angles right angles.



Let ABCD be a rectangle having A a right angle. Join BD. Then in the triangles ABD, CBD we have  $AB=CD$ ,  $AD=BC$ , and the base BD common;

$\therefore$  the angle BCD = the angle BAD = a right angle, and the angle ABD = the angle BDC, so that AB, CD are parallel, and the angle ADB = the angle DBC, so that AD, BC are parallel.

Since AD, BC are parallel, the angles DAB, ABC are equal to two right angles, of which DAB is a right angle;

$\therefore$  the angle ABC is a right angle.

Similarly ADC is a right angle.

**EXERCISES.**

1. Prove that the sides of two equal squares must be equal.
2. The square on a given straight line is four times the square on half the line.
3. If, in the sides of AB, BC, CD, DA of a square ABCD points E, F, G, H be taken so that AE, BF, CG, DH are equal, then EFGH is a square.
4. If the diagonals of a quadrilateral are equal and bisect each other at right angles, the quadrilateral is a square.
5. On the sides AC, BC of a triangle ABC, squares ACDE, BCFH are described: shew that the straight lines AF and BD are equal.
6. Squares are described on the three sides of any triangle, the squares being all outside the triangle, and adjacent corners of the squares are joined; shew that the triangles so formed are each equal in area to the original triangle.

[Let ABC be the  $\triangle$ , and BDEC, CFGA, AHKB the squares on its sides; since BCE, ACF are right  $\angle^s$ ,  $\therefore$  ECF, ACB are supplementary angles [I. 15, Cor. 2];  $\therefore$  ACB, ECF are two  $\triangle^s$  with two sides equal and the included  $\angle^s$  supplementary. Now use Ex. 3 on Prop. 38.]

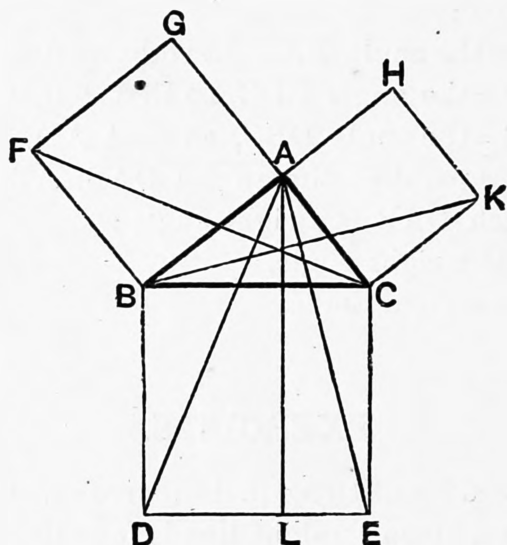


## PROPOSITION 47. THEOREM.

*In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.*

Let  $ABC$  be a right-angled triangle, having the right angle  $BAC$ ;

*the square described on the side  $BC$  shall be equal to the squares described on the sides  $BA$ ,  $AC$ .*



**Construction.** On  $BC$  describe the square  $BDEC$ , and on  $BA$ ,  $AC$  describe the squares  $BFGA$ ,  $AHKC$ ; [I. 46.  
through  $A$  draw  $AL$  parallel to  $BD$  or  $CE$ , [I. 31.  
and join  $AD$ ,  $FC$ .

**Proof.** Because the angle  $BAC$  is a right angle, [*Hypothesis*.  
and that the angle  $BAG$  is also a right angle; [*Definition* 31.  
 $\therefore$   $CA$  is in the same straight line with  $AG$ . [I. 14.

Similarly,  $AB$  and  $AH$  are in the same straight line.

Now the angle  $DBC =$  the angle  $FBA$ ,

for each of them is a right angle; [*Axiom* 11.

add to each the angle  $ABC$ .

$\therefore$  the whole angle  $DBA =$  the whole angle  $FBC$ . [*Axiom* 2.

Then, in the triangles DBA, FBC,

because  $\left\{ \begin{array}{l} AB = FB, \\ \text{and } BD = BC, \\ \text{and the angle } ABD = \text{the angle } FBC, \end{array} \right. \begin{array}{l} [Constr. \\ [Constr. \\ [Proved. \end{array}$

$\therefore$  the  $\triangle ABD =$  the  $\triangle FBC$ . [I. 4.

Now the parallelogram BL is double of the  $\triangle ABD$ , because they are on the same base BD, and between the same parallels BD, AL. [I. 41.

And the square GB is double of the  $\triangle FBC$ , because they are on the same base FB, and between the same parallels FB, GC. [I. 41.

But the doubles of equals are equal. [Axiom 6.

$\therefore$  the parallelogram BL = the square GB.

Similarly, by joining AE, BK, it can be shewn that the parallelogram CL = the square CH.

$\therefore$  the whole square BDEC = the two squares GB, HC. [Ax. 2.

And the square BDEC is described on BC, and the squares GB, HC on BA, AC.

$\therefore$  the square described on the side BC  
= the squares described on the sides BA, AC.

Wherefore, *in any right-angled triangle, etc.* [Q.E.D.

#### NOTE ON I. 47.

Tradition ascribed the discovery of I. 47 to Pythagoras, who flourished about 570 to 500 B.C. Many demonstrations have been given of this celebrated proposition; the following is one of the most interesting:

Let ABCD, AEFG be any two squares, placed so that their bases



**6.** The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

**7.** If  $ABC$  be a triangle whose angle  $A$  is a right angle, and  $BE$ ,  $CF$  be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on  $BE$  and  $CF$  is equal to five times the square on  $BC$ .

**\*\*8.** The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

[Let  $ABC$  be the  $\triangle$ ,  $C$  being acute; make  $BCD$  a rt.  $\angle$  and  $CD = CA$ . Then, by I. 24,  $BA < BD \therefore BA^2 < BD^2$ , i.e.  $< BC^2 + CD^2$ , i.e.  $< BC^2 + CA^2$ .]

**\*\*9.** The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

**\*\*10.** If the square on one side of a triangle be less than the squares on the other two sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle.

In the figure of I. 47, prove that

**11.**  $AD$  and  $FC$  are at right angles.

[Let  $AD$  meet  $FC$  in  $O$  and  $BC$  in  $V$ .

Then  $\angle AOC = \angle OVC + \angle OCV = \angle BVD + \angle BDV = \text{a rt. } \angle$ .]

**12.**  $F$ ,  $A$ ,  $K$  are in a straight line.

**13.**  $FG$ ,  $HK$ , and  $AL$  meet in a point.

[Let  $FG$  meet  $KH$  in  $U$ ; then  $\triangle^s AHU$ ,  $CAB$  are equal in all respects, so that  $\angle HAU = \angle ACB = \text{complement of } \angle ABC = \angle BAL$ , etc.]

**14.**  $BG$ ,  $CH$  are parallel.

**15.** If  $DM$ ,  $EN$  are drawn perpendicular to  $AC$ ,  $AB$ , then  $AM$  equals  $AB$ , and  $AN$  equals  $AC$ .

[Draw  $DT$  perp<sup>r</sup> to  $AB$  produced; then  $DTAM$  is a rectangle and  $\therefore AM = DT$ . It can then be proved that  $\triangle^s DTB$ ,  $BAC$  are equal in all respects, and  $\therefore DT = AB$ .]

**16.** Divide a straight line into two parts the sum of the squares on which is equal to a given square. What limit is there to the size of this given square?

[ $AB$  being given, make  $\angle ABC = \text{half a rt. } \angle$ . With centre  $A$  and radius = side of given square describe a circle to meet  $BC$  in  $P$ . Draw  $PM$  perp<sup>r</sup> to  $AB$ . Then  $AB$  is divided as required in  $M$ .]

**17.** In a straight line  $AB$ , produced if necessary, find a point  $D$  such that the difference of the squares on  $AD$ ,  $BD$  may be equal to a given square. [See App. Art. 17.]

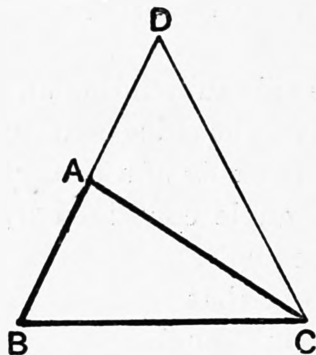
**18.** Shew that a right-angled triangle can be made whose sides are proportional to the numbers 3, 4, and 5.

## PROPOSITION 48. THEOREM.

*If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.*

Let the square described on BC, one of the sides of the triangle ABC, be equal to the squares described on the other sides BA, AC:

*the angle BAC shall be a right angle.*



**Construction.** From the point A draw AD at right angles to AC ; [I. 11.]

make AD equal to BA, and join DC.

**Proof.** Because DA = BA, the square on DA = the square on BA. To each of these add the square on AC.

$\therefore$  the squares on DA, AC = the squares on BA, AC. [Axiom 2.]

But because the angle DAC is a right angle, [Construction.]

the square on DC = the squares on DA, AC ; [I. 47.]

and the square on BC = the squares on BA, AC ; [Hypothesis.]

$\therefore$  the square on DC = the square on BC ; [Axiom 1.]

$\therefore$  DC = BC.

Then, in the triangles BAC, DAC,

because  $\begin{cases} BA = AD, \\ \text{and } AC \text{ is common,} \\ \text{and the base } BC = \text{the base } CD ; \end{cases}$

[Construction.]

$\therefore$  the angle DAC = the angle BAC.

[I. 8.]

But DAC is a right angle ;

[Construction.]

$\therefore$  also BAC is a right angle.

[Axiom 1.]

Wherefore, *if the square, etc.*

[Q. E. D.]

## BOOK II.

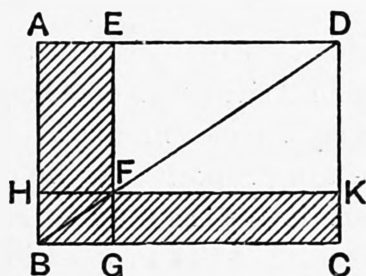
### DEFINITIONS.

**1.** Every right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

Thus the rectangle  $ABCD$  is said to be *contained* by the straight lines  $AB$  and  $AD$ , or by the straight lines  $BA$ ,  $BC$ , etc.

We shall use the abbreviations “the rectangle  $AB$ ,  $BC$ ,” or “the rect.  $AB$ ,  $BC$ ” for the expression “the rectangle contained by  $AB$ ,  $BC$ .”

**2.** In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.

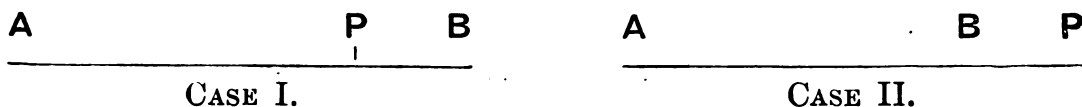


Thus the parallelogram  $HG$ , together with the complements  $AF$ ,  $FC$ , is the gnomon, which is more briefly expressed by the letters  $AGK$ , or  $EHC$ , which are at the opposite angles of the parallelograms which make the gnomon.

Similarly, the figure consisting of  $EK$  together with the complements  $AF$ ,  $FC$  (that is, all the figure except  $HG$ ) is called the gnomon  $AKG$ .

**3.** When a straight line is divided into two parts, each part is called a segment by Euclid. It is found convenient to extend the meaning of the word *segment*, and to lay down the following definition: When a point  $P$  is taken in a straight line  $AB$ , or in the straight line  $AB$  produced, its distances

from the ends of the straight line are called segments of the straight line. When it is necessary to distinguish them, such segments are called **internal** or **external**, according as the point is in the straight line, or in the straight line produced.



### NOTE.

There is an analogy between the first ten propositions of this book and some elementary facts in Arithmetic and Algebra.

It is shewn in Arithmetic that if one side of a rectangle contains a unit of length an exact number of times, and if an adjacent side also contains the same unit an exact number of times, the product of these units will be the number of square units in the area of the rectangle.

Thus, if the sides be respectively  $m$  inches and  $n$  inches, the area is  $mn$  square inches.

Similarly, if a square have each of its sides equal to  $m$  units, its area is  $m^2$  square units.

We thus see that the area of a rectangle in geometry corresponds to a product of two numbers in Arithmetic or Algebra; whilst the area of a square corresponds to a square of a number.

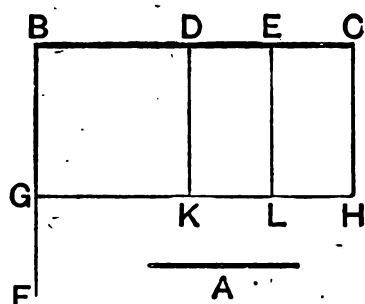
We shall add to these ten propositions the corresponding algebraic formulae. By means of the latter the student is enabled to more easily keep in his memory the results of the Propositions. The proofs in the Propositions are, however, more general than those of the formulae, since in Geometry it is not assumed that the lines spoken of are *commensurable*. We do not enter on this subject as it would lead us too far from Euclid's *Elements of Geometry* with which we are here occupied.

Owing to the above analogy the expression "the square on AB" is often abbreviated into " $AB^2$ ," and "the rectangle AB, BC" into " $AB \cdot BC$ ." We shall sometimes use these abbreviations in the course of this book, and the student may use them in Exercises. We shall not use them in the text of the Propositions, nor should the student do so in writing out the Proposition in an Examination. The signs  $+$  and  $-$  may be used in Deductions. But the student must always carefully note that such an expression as " $AB^2 + BC \cdot CD$ " is only an *abbreviation* for "the square on AB together with the rectangle BC, CD."

## PROPOSITION 1. THEOREM.

*If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.*

Let  $A$  and  $BC$  be two straight lines; and let  $BC$  be divided into any number of parts at the points  $D, E$ :  
*the rectangle contained by  $A, BC$  shall be equal to the sum of the rectangles contained by  $A, BD$ , by  $A, DE$ , and by  $A, EC$ .*



**Construction.** From  $B$  draw  $BF$  at right angles to  $BC$ , [I. 11.  
 and make  $BG$  equal to  $A$ ; [I. 3.  
 through  $G$  draw  $GH$  parallel to  $BC$ ; and through  $D, E, C$   
 draw  $DK, EL, CH$ , parallel to  $BG$ . [I. 31.]

**Proof.** The rectangle  $BH$  = the sum of the rectangles  $BK, DL, EH$ .

But  $BH$  is contained by  $A, BC$ , for it is contained by  $GB, BC$ , and  $GB = A$ . [Construction.]

And  $BK$  is contained by  $A, BD$ , for it is contained by  $GB, BD$ , and  $GB = A$ ;

and  $DL$  is contained by  $A, DE$ , because  $DK$  is equal to  $BG$ , which is equal to  $A$ ; [I. 34.]

and in like manner  $EH$  is contained by  $A, EC$ ;

$\therefore$  the rectangle contained by  $A, BC$  = the sum of the rectangles contained by  $A, BD$ , and by  $A, DE$ , and by  $A, EC$ .

Wherefore, *if there be two straight lines, etc.* [Q.E.D.]

**Algebraic Formula.**  $a(b + c + d) = ab + ac + ad$ .

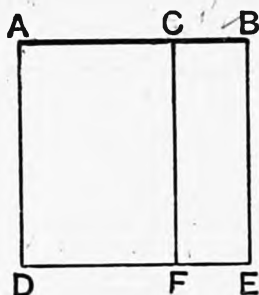


## PROPOSITION 2. THEOREM.

*If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.*

Let the straight line AB be divided into any two parts at the point C :

*the rectangle AB, BC, together with the rectangle AB, AC, shall be equal to the square on AB.*



**Construction** On AB describe the square ADEB ; [I. 46. and through C draw CF parallel to AD or BE. [I. 31.

**Proof.** AE is equal to the rectangles AF, CE.  
But AE is the square on AB.

Also AF is the rectangle contained by BA, AC ;  
for it is contained by DA, AC, of which DA = BA ;  
and CE is contained by AB, BC, for BE = AB ;  
∴ the rectangle AB, AC, together with the rectangle AB, BC,  
= the square on AB.

Wherefore, *if a straight line, etc.* [Q. E. D.

**Algebraic Formula.** Let AC be  $a$  units, and CB be  $b$  units ; then

$$a(a + b) + b(a + b) = (a + b)^2$$

## EXERCISE.

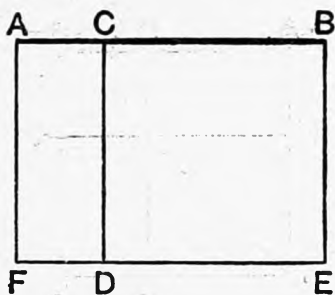
If a straight line be divided internally into any number of segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the several segments.

## PROPOSITION 3. THEOREM.

*If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.*

Let the straight line AB be divided into any two parts at the point C :

*the rectangle AB, BC shall be equal to the square on BC, together with the rectangle AC, CB.*



**Construction.** On BC describe the square CDEB ; [I. 46. produce ED to F, and through A draw AF parallel to CD or BE. [I. 31.

**Proof.** The rectangle AE = the rectangles AD, CE.  
But AE is the rectangle contained by AB, BC ; for it is contained by AB, BE, of which BE = BC ;  
and AD is contained by AC, CB, since CD = CB ;  
and CE is the square on BC ;

the rectangle AB, BC = the square on BC, together with the rectangle AC, CB.

Wherefore, *if a straight line, etc.*

[Q. E. D.

**Algebraic Formula.** Let AC be  $a$  units, and CB be  $b$  units ; then

$$(a + b)b = ab + b^2.$$

## EXERCISE.

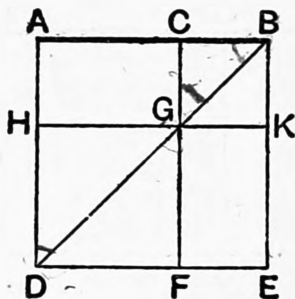
A, B, C, D are four points in a straight line taken in order ; prove that the rectangle AC, BD is equal to the sum of the rectangles AB, CD and AD, BC.

## PROPOSITION 4. THEOREM.

*If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.*

Let the straight line  $AB$  be divided into any two parts at the point  $C$  :

*the square on  $AB$  shall be equal to the squares on  $AC$ ,  $CB$ , together with twice the rectangle  $AC$ ,  $CB$ .*



**Construction.** On  $AB$  describe the square  $ADEB$ ; [I. 46. join  $BD$ ; through  $C$  draw  $CGF$  parallel to  $AD$  or  $BE$ , and through  $G$  draw  $HGK$  parallel to  $AB$  or  $DE$ . [I. 31.

**Proof.** Because  $CF$  is parallel to  $AD$ , and  $BD$  meets them, the exterior angle  $CGB$  = the interior and opposite angle  $ADB$ ; [I. 29.

but the angle  $ADB$  = the angle  $ABD$ , [I. 5.

because  $BA = AD$ , being sides of a square,

$\therefore$  the angle  $CGB$  = the angle  $CBG$ ; [Axiom 1.

$\therefore CG = CB$ . [I. 6.

But  $CB = GK$ , and  $CG = BK$ ; [I. 34.

$\therefore CK$  is equilateral.

Also,  $CBK$  is a right angle; [I. 46.

$\therefore CK$  is a parallelogram with all its sides equal, and one angle a right angle;

$\therefore$  it is square, and it is on the side  $CB$ . [Definition 31.

Similarly,  $HF$  is the square on  $HG$  which =  $AC$ . [I. 34.

$\therefore HF$ ,  $CK$  are the squares on  $AC$ ,  $CB$ .

Now, the complement  $AG$  = the complement  $GE$ ; [I. 43.  
 and  $AG$  is the rectangle  $AC, CB$ , since  $CG = CB$ ;  
 $\therefore GE$  also = the rectangle  $AC, CB$ ; [Ax. 1.  
 $\therefore AG, GE$  = twice the rectangle  $AC, CB$ .

Finally, the square on  $AB$  = the figure  $AE$ ,  
 that is, = the four figures  $HF, CK, AG, GE$ ,  
 that is, = the squares on  $AC, CB$ , together with twice the  
 rectangle  $AC, CB$ .

Wherefore, *if a straight line, etc.* [Q.E.D.

**Corollary 1.** Parallelograms about the diameter of a square  
 are likewise squares.

**Corollary 2.** If, in the previous proposition,  $C$  bisect  $AB$ ,  
 the rectangle  $AC, CB$  is equal to the square on  $AC$ , so that  
 the proposition states that, in this case, the square on  $AB$  is  
 four times the square on  $AC$ , that is,  
*the square on twice a given line is four times the square on the line.*

**Algebraic Formula.** Let  $AC$  be  $a$  units, and  $CB$  be  $b$   
 units; then  $(a + b)^2 = a^2 + b^2 + 2ab$ .

### ALTERNATIVE PROOF.

Because  $AB$  is divided into two parts at  $C$ ;

$\therefore$  square on  $AB$  = sum of rectangles  $AB, AC$  and  $AB, CB$ . [II. 2.

Also, by II. 3,

the rectangle  $AB, AC$  = square on  $AC$ , together with rect.  $AC, CB$ ;  
 and the rectangle  $AB, CB$  = square on  $CB$ , together with rect.  $AC, CB$ ;  
 $\therefore$  the square on  $AB$  = the squares on  $AC, CB$ , together with twice the  
 rectangle  $AC, CB$ .

### EXERCISES.

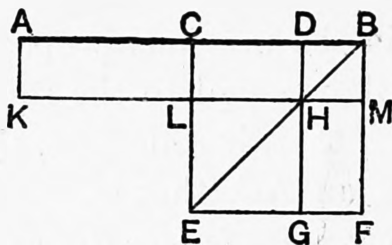
**1.** Prove the truth of II. 4 as follows:—On the sides  $AD, DE, EB$   
 of the square  $ADEB$  take points  $P, Q, R$  such that  $AP, DQ$ , and  $ER$   
 are each equal to  $BC$ ; shew that the four triangles  $APC, DQP, ERQ$   
 and  $BCR$  are each one-half the rect.  $AC, CB$ , and that  $PQRC$  is the  
 square on  $CP$ , which equals the squares on  $AP, AC$ , that is, on  $BC, AC$ .

**2.** In the figure of I. 1 if the circles meet again at  $F$  prove that the  
 square on  $CF$  is three times the square on  $AB$ .

## PROPOSITION 5. THEOREM.

*If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.*

Let the straight line  $AB$  be divided into two equal parts at the point  $C$ , and into two unequal parts at the point  $D$  :  
*the rectangle  $AD, DB$ , together with the square on  $CD$ , shall be equal to the square on  $CB$ .*



**Construction.** On  $CB$  describe the square  $CEFB$ ; [I. 46. join  $BE$ ; through  $D$  draw  $DHG$  parallel to  $CE$  or  $BF$ ; through  $H$  draw  $KLM$  parallel to  $CB$  or  $EF$ ; and through  $A$  draw  $AK$  parallel to  $CL$  or  $BM$ . [I. 31.

**Proof.** The complement  $CH$  = the complement  $HF$ ; [I. 43. to each of these add  $DM$ ;

$\therefore$  the whole  $CM$  = the whole  $DF$ . [Axiom 2.

But  $CM = AL$ , because  $AC = CB$ ; [I. 36.

$\therefore$  also  $AL = DF$ ; to each add  $CH$ ;

$\therefore$  the whole  $AH = DF$  and  $CH$ . [Axiom 2.

But  $AH$  is the rectangle  $AD, DB$ , for  $DH = DB$ ; [II. 4, Corollary 1. and  $DF$  together with  $CH$  is the gnomon  $CMG$ ; therefore the rectangle  $AD, DB$  = the gnomon  $CMG$ .

To each add the square on  $CD$ , which =  $LG$ ; [II. 4, Cor. 1, and I. 34.

$\therefore$  the rectangle  $AD, DB$ , together with the square on  $CD$ , = the gnomon  $CMG$ , together with  $LG$ ,

that is, = the figure  $CEFB$ , which is the square on  $CB$ .

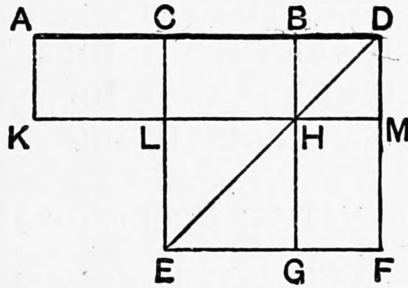
Wherefore, if a straight line, etc.

[Q.E.D.]

## PROPOSITION 6. THEOREM.

*If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.*

Let the straight line  $AB$  be bisected at the point  $C$ , and produced to the point  $D$  :  
*the rectangle  $AD$ ,  $DB$ , together with the square on  $CB$ , shall be equal to the square on  $CD$ .*



**Construction.** On  $CD$  describe the square  $CEFD$  ; [I. 46.  
 join  $DE$  ; through  $B$  draw  $BHG$  parallel to  $CE$  or  $DF$  ; through  
 $H$  draw  $KLM$  parallel to  $AD$  or  $EF$  ; and through  $A$  draw  $AK$   
 parallel to  $CL$  or  $DM$ . [I. 31.]

**Proof.** Because  $AC = CB$ , [Hypothesis.  
 the rectangle  $AL =$  the rectangle  $CH$  ; [I. 36.  
 but the complement  $CH =$  the complement  $HF$  ; [I. 43.  
 $\therefore$  also  $AL = HF$ . [Axiom 1.]

To each add  $CM$  ;

$\therefore$  the whole  $AM =$  the gnomon  $CMG$ . [Axiom 2.]

But  $AM$  is the rectangle contained by  $AD$ ,  $DB$ , since  $DM = DB$  ;  
 [II. 4, Corollary 1.]

$\therefore$  the rectangle  $AD$ ,  $DB =$  the gnomon  $CMG$ . [Axiom 1.]

To each add the square on  $CB$ , which  $= LG$  ; [II. 4, Cor. 1, and I. 34.]

$\therefore$  the rectangle  $AD$ ,  $DB$ , together with the square on  $CB$ ,  
 $=$  the gnomon  $CMG$  and the figure  $LG$ ,

that is,  $=$  the figure  $CEFD$ , which is the square on  $CD$ .

Wherefore, if a straight line, etc.

[Q. E. D.]

**Algebraic Formula for Prop. 5.** Let AC or CB be  $a$  units and CD be  $b$  units, so that

$$AD = (a + b) \text{ units and } DB = (a - b) \text{ units ;}$$

then

$$(a + b)(a - b) + b^2 = a^2.$$

**Algebraic Formula for Prop. 6.** Let AC = CB =  $a$  units and CD =  $b$  units ; then

$$(a + b)(b - a) + a^2 = b^2.$$

#### ALTERNATIVE PROOF OF II. 5.

The sq. on CB = sqs. on CD, DB and twice the rect. CD, DB [II. 4.  
 = sq. on CD with rect. CD, DB and rect. DB, CB [II. 3.  
 = sq. on CD with rect. CD, DB and rect. AC, DB  
 = sq. on CD and rect. AD, DB. [II. 1.

#### ALTERNATIVE PROOF OF II. 6.

The sq. on CD = sqs. on CB, BD with twice the rect. CB, BD [II. 4.  
 = sq. on CB with rect. CB, BD and rect. CD, BD [II. 3.  
 = sq. on CB with rect. AC, BD and rect. CD, BD  
 = sq. on CB with rect. AD, DB. [II. 1.

#### NOTE ON II. 5.

From this proposition it is clear that *the difference of the squares on two unequal straight lines AC, CD is equal to the rectangle contained by their sum and difference.*

For the proposition states that the difference between the squares on AC, CD = the rect. AD, DB.

Also AD = the sum of AC and CD, and DB = difference between CB and CD = the diff. between AC and CD.

Again, the proposition says that *the rectangle contained by two straight lines is equal to the difference between the squares on their semi-sum and their semi-difference.*

For the rect. AD, DB = difference of sqs. on CB, CD.

Also CB = half of AB = half the sum of AD, DB,  
 and CD = half the difference of AD and DB,

since  $AD - DB = AC + CD - DB = CB + CD - DB = 2CD.$

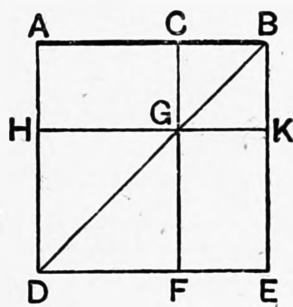
With a slight alteration of lettering the above is also true for the figure of II. 6.

## PROPOSITION 7. THEOREM.

*If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.*

Let the straight line AB be divided into any two parts at the point C :

*the squares on AB, BC shall be equal to twice the rectangle AB, BC, together with the square on AC.*



**Construction.** On AB describe the square ADEB, and construct the figure as in the preceding propositions.

**Proof.** The complement AG = the complement GE ; [I. 43.  
to each of these add CK ;

$\therefore$  the whole AK = the whole CE ;

$\therefore$  AK, CE are double of AK.

But twice the rect. AB, BC is double of AK,

since BK = BC.

[II. 4, Cor. 1.

$\therefore$  twice the rectangle AB, BC = the figures AK, CE,

that is, = gnomon AKF, together with the square CK.

To each add the square on AC, which = HF ; [II. 4, Cor. 1 and I. 34.

$\therefore$  twice the rectangle AB, BC, together with the square on AC,

= the gnomon AKF, together with the squares CK, HF,

that is, = the whole figure ADEB, together with CK,

that is, = the squares on AB, BC.

Wherefore, if a straight line, etc.

[Q.E.D.



**Algebraic Formula.** Let AB be  $a$  units and CB be  $b$  units; then  $a^2 + b^2 = 2ab + (a - b)^2$ .

### ALTERNATIVE PROOF.

Since AB is divided into two parts at C,  
 $\therefore$  sq. on AB = sqs. on AC, CB, and twice the rectangle AC, CB. [II. 4.  
 To each add the square on CB;

$\therefore$  the squares on AB, CB = the square on AC, together with twice the square on CB, and twice the rectangle AC, CB.

But the rect. AB, BC = sq. on CB, and the rect. AC, CB; [II. 3.  
 $\therefore$  twice the rect. AB, BC = twice the square on CB and twice the rect. AC, CB; [Axiom 6.

$\therefore$  the squares on AB, CB = the square on AC, together with twice the rectangle AB, BC.

### NOTE TO PROPOSITION 7.

This proposition may be enunciated thus:

*The square described on a straight line which is equal to the difference of two straight lines is less than the sum of the squares on the two straight lines by twice the rectangle contained by them.*

### EXERCISES.

**\*\*1.** Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

[In the figure to II. 5 the rect. AD, DB is greatest when CD vanishes, i.e. when D is at C.]

**2.** The least square that can be inscribed in a given square is that which is one half the given square.

[Let ABCD be the given square, and E, F, G, H points in the sides such that AE = BF = CG = DH; then the  $\triangle$ 's AEH, EBF, FCG, GDH are all equal;  $\therefore$  the sq. EFGH is least when the  $\triangle$  AEH is greatest, i.e. when the rect. AE, AH is greatest (I. 41), i.e. when AE, EB is greatest, i.e. by Ex. 1, when E is the middle point of AB;  $\therefore$  etc.]

**\*\*3.** Of all rectangles with the same perimeter, the square has the greatest area. [Use Ex. 1.]

**\*\*4.** If ABC is an isosceles triangle and D any point on the base BC, prove that the rectangle BD, DC is equal to the difference of the squares on AC, AD.

**5.** The square on either of the sides about the right angle of a right-angled triangle is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other side.

[Use I. 47 and the Note to II. 5.]

**6.** Construct a rectangle equal to the difference of two given squares. [Use II. 5 or 6.]

**\*\*7.** ABC is a triangle having a right angle at A, and AD is the perpendicular on BC; prove that

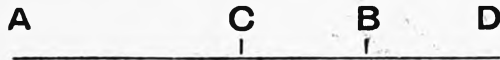
(1)  $AD^2 = \text{rect. BD, DC}$ , and (2)  $AC^2 = \text{rect. BC, CD}$ .

## PROPOSITION 8. THEOREM.

*If a straight line be divided into any two parts, four times the rectangle contained by the whole and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.*

[Euclid's proof of this proposition is cumbrous; it is seldom read or required. The following proof is easy.]

Let the straight line AB be divided into two parts at B :  
*then four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the straight line made up of AB and BC together.*



**Construction.** Produce AB to D, making BD equal to CB.

**Proof.** Since AD is divided into two parts at B ;  
 $\therefore$  the square on AD = the squares on AB, BD, together with twice the rect. AB, BD. [II. 4.  
 But sq. on BD = sq. on BC, and rect. AB, BD = rect. AB, BC,  
 since BD = CB ;  
 $\therefore$  sq. on AD = sqs. on AB, BC and twice the rect. AB, BC  
 But by II. 7,  
 the sqs. on AB, BC = sq. on AC and twice the rect. AB, BC ;  
 $\therefore$  sq. on AD = sq. on AC, and four times the rect. AB, BC.  
 Also AD is made up of AB and CB together, since BD = CB.  
 Wherefore, *etc.*

**Algebraic Formula.** Let  $AB = a$  units, and  $CB = BD = b$  units ; then  $4ab + (a - b)^2 = (a + b)^2$ .

## NOTE TO PROPOSITION 8.

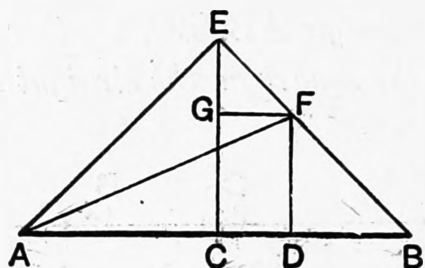
Since AD is the sum of the two lines AB, BC, and AC is the difference of the same two lines, this proposition may be enunciated thus :

*The square on the sum of two straight lines = the square on the difference of the two lines, together with four times the rect. contained by the two lines.*

## PROPOSITION 9. THEOREM.

*If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.*

Let the straight line  $AB$  be divided into two equal parts at the point  $C$ , and into two unequal parts at the point  $D$ :  
the squares on  $AD$ ,  $DB$  shall be together double of the squares on  $AC$ ,  $CD$ .



**Construction.** From  $C$  draw  $CE$  at right angles to  $AB$ , [I. 11.  
and make it equal to  $AC$  or  $CB$ , [I. 3.

and join  $EA$ ,  $EB$ ;

draw  $DF$  parallel to  $CE$  to meet  $BE$  in  $F$ ,

and  $FG$  parallel to  $BA$  to meet  $CE$  in  $G$ , [I. 31.

and join  $AF$ .

**Proof.** (1) Because  $ACE$  is a right angle, [Construction.  
the two other angles  $AEC$ ,  $EAC$  are together equal to one  
right angle; [I. 32.

and they are equal, since  $CE$  was made equal to  $AC$ ; [I. 5.

$\therefore$  each of them is half a right angle.

Similarly, each of the angles  $CEB$ ,  $EBC$  is half a right angle;

$\therefore$  the whole angle  $AEB$  is a right angle.

(2) Because the angle  $GEF$  is half a right angle, and the  
angle  $EGF$  a right angle,

for it = the interior and opposite angle  $ECB$ ; [I. 29.

$\therefore$  the remaining angle  $EFG$  is half a right angle;

$\therefore$  the angle  $GEF$  = the angle  $EFG$ , and  $EG = GF$ . [I. 6.

(3) Again, because the angle at B is half a right angle, and FDB a right angle,

for it = the interior and opposite angle ECB ; [I. 29.

∴ the remaining angle BFD is half a right angle ; [I. 32.

∴ the angle at B = the angle BFD, and DF = DB. [I. 6.

(4) The square on AE = the squares on AC, CE, [I. 47.

that is, = twice the square on AC, since CE = AC. [Constr.

Again, the square on EF = the squares on EG, GF, [I. 47.

that is, = twice the square on GF, since EG = GF ; [Proved.

that is, = twice the square on CD, since GF = CD. [I. 34.

∴ twice the squares on AC, CD = the sqs. on AE, EF,

that is, = the square on AF, since AEF is a right angle, [I. 47.

that is, = the squares on AD, DF, since ADF is a rt. angle,

[I. 47.

that is, = the sqs. on AD, DB, since DB = DF.

Wherefore, *if a straight line, etc.*

[Q.E.D.

**Algebraic Formula.** Let  $AC = CB = a$  units and  $CD = b$  units ; then

$$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2.$$

### ALTERNATIVE PROOF OF II. 9.

Since AD is divided at C,

∴ sq. on AC, CD + twice the rect. AC, CD = sq. on AD. [II. 4.

Again, since BC is divided internally at D,

∴ sqs. on BC, CD = twice the rect. BC, CD + sq. on BD ; [II. 7.

that is, sqs. on AC, CD = twice the rect. AC, CD + sq. on BD,

since AC = BC ; add these results ;

∴ twice the squares on AC, CD, together with twice the rect. AC, CD

= the squares on AD, BD, together with twice the rect. AC, CD ;

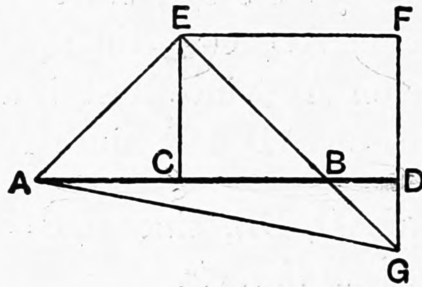
∴ twice the squares on AC, CD alone = the squares on AD, BD.

[If in the third line of this proof we read “externally” instead of “internally,” it will apply also to Prop. 10.]

## PROPOSITION 10. THEOREM.

*If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.*

Let the straight line  $AB$  be bisected at  $C$ , and produced to  $D$ : the squares on  $AD$ ,  $DB$  shall be together double of the squares on  $AC$ ,  $CD$ .



**Construction.** From  $C$  draw  $CE$  at right angles to  $AB$ ,  
[I. 11.]  
and make it equal to  $AC$  or  $CB$ ;  
[I. 3.]  
and join  $AE$ ,  $EB$ ;  
draw  $EF$  parallel to  $AB$ , and  $DF$  parallel to  $CE$ . [I. 31.]

**Proof.** (1) Because  $EF$  meets the parallels  $EC$ ,  $FD$ , the angles  $CEF$ ,  $EFD$  are together equal to two right angles; [I. 29.]  
 $\therefore$   $BEF$ ,  $EFD$  are together less than two right angles;  
 $\therefore$   $EB$ ,  $FD$  will meet, if produced, towards  $B$ ,  $D$ . [Axiom 12.]

Let them meet at  $G$ , and join  $AG$ .

Then, because  $ACE$  is a right angle; [Construction.]  
the angles  $AEC$ ,  $EAC$  together = a rt. angle, [I. 32.]  
and they are equal, since  $AC = CE$ ; [Construction.]  
 $\therefore$  each of the angles  $CEA$ ,  $EAC$  is half a right angle. [I. 32.]  
Similarly, each of the angles  $CEB$ ,  $EBC$  is half a right angle;  
 $\therefore$   $AEB$  is a right angle.

(2) And because  $EBC$  is half a right angle, the vertically opposite angle  $DBG$  is also half a right angle; [I. 15.]

but  $\angle BDG$  is a right angle, because it is equal to the alternate angle  $\angle DCE$ ; [I. 29.]

$\therefore$  the remaining angle  $\angle DGB$  is half a right angle, [I. 32.]

and is therefore equal to the angle  $\angle DBG$ ;

$\therefore$  also the side  $BD =$  the side  $DG$ . [I. 6.]

(3) Again, because  $\angle EGF$  is half a right angle, and the angle at  $F$  a right angle, for it  $=$  the angle  $\angle ECD$ ; [I. 34.]

$\therefore$  the remaining angle  $\angle FEG$  is half a right angle, [I. 32.]

and is therefore equal to the angle  $\angle EGF$ ;

$\therefore$  also  $GF = FE$ . [I. 6.]

(4) The square on  $AE =$  the squares on  $EC, CA$ , [I. 47.]

that is,  $=$  twice the square on  $AC$ ,

since  $EC = AC$ . [Constr.]

Also the square on  $EG =$  the squares on  $GF, FE$ , [I. 47.]

that is,  $=$  twice the square on  $FE$ ,

since  $GF = FE$ , [Proved in (3).]

that is,  $=$  twice the square on  $CD$ . [I. 34.]

Hence, twice the sqs. on  $AC, CD =$  the sqs. on  $AE, EG$ ,

that is,  $=$  the sq. on  $AG$ , since  $\angle AEG$  is a rt. angle, [I. 47.]

that is,  $=$  the sqs. on  $AD, DG$ , since  $\angle ADG$  is a rt. angle, [I. 47.]

that is,  $=$  the sqs. on  $AD, DB$ , since  $DB = DG$ . [Proved in (2).]

Wherefore, *if a straight line, etc.* [Q. E. D.]

**Algebraic Formula.** Let  $AC = CB = a$  units and  $CD = b$  units; then  $(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$ .

[For Exercises, see Page 105.]

## NOTE ON PROPOSITIONS 9 AND 10.

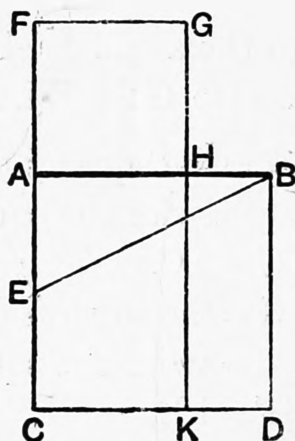
These two propositions may be enunciated in one thus:

*The sum of the squares on the sum and difference of two lines = twice the sum of the squares on the two lines.* For in each case  $AD$  is the sum, and  $DB$  the difference of the two lines  $AC, CD$ .

## PROPOSITION 11. PROBLEM.

*To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.*

Let  $AB$  be the given straight line :  
*it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.*



**Construction.** On  $AB$  describe the square  $ABDC$ ; [I. 46.  
 bisect  $AC$  at  $E$ ; [I. 10.  
 join  $BE$ ; produce  $CA$  to  $F$ , and make  $EF$  equal to  $EB$ ; [I. 3.  
 on  $AF$  describe the square  $AFGH$ . [I. 46.  
 $AB$  shall be divided at  $H$  so that the rectangle  $AB, BH$  is  
 equal to the square on  $AH$ .

Produce  $GH$  to meet  $CD$  in  $K$ .

**Proof.** Because  $AC$  is bisected at  $E$ , and produced to  $F$ ,  
 the rectangle  $CF, FA$ , together with the square on  $AE$ ,  
     = the square on  $EF$ ; [II. 6.  
 that is, = the square on  $EB$ , since  $EF = EB$ . [Construction.  
 that is, = the sqs. on  $AE, AB$ , since  $EAB$  is a rt. angle; [I. 47.  
 From each of these equals take the square on  $AE$   
 $\therefore$  the rect.  $CF, FA$  = the square on  $AB$ . [Axiom 3.  
 But  $FK$  is the rectangle  $CF, FA$ ; for  $FG = FA$ , [Constr.  
 and  $AD$  is the square on  $AB$ ;  
 $\therefore FK = AD$ .

Take away the common part AK,

and the remainder FH = the remainder HD.

[Axiom 3.

But HD is the rectangle AB, BH ; for AB = BD ;

[Constr.

and FH is the square on AH ;

[Constr.

$\therefore$  the rectangle AB, BH = the square on AH.

Wherefore *the straight line AB is divided at H, so that the rectangle AB, BH is equal to the square on AH.*

[Q.E.F.

*Note.* When a straight line AB is divided as in the above proposition, it is said to be divided in **medial section**

### EXERCISES ON PROPOSITIONS 9 and 10.

1. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

[In the figure of II. 9,  $AD^2 + DB^2$  is least when CD vanishes, that is, when D is the middle point of AB.]

2. The sum of the squares on two straight lines is never less than twice the rectangle contained by them and is never less than half the square on their sum. [Use Propositions 7 and 9.]

3. If AB be bisected in C and divided unequally at D, then the sum of the squares on AD, DB is equal to twice the rectangle AD, DB together with four times the square on CD. [Use Propositions 9 and 5.]

4. C is the middle point of AB and D any point on AB produced ; prove that the square on AD is equal to the square on BD, together with four times the rectangle AC, CD.

### EXERCISES ON PROPOSITION 11.

**\*\*1.** *Divide a given straight line externally in medial section.*

Let AB be the given straight line. On it describe the square ACDB. Bisect AC in E and produce EC to F, making EF equal to EB. On AF, on the side remote from BD, describe the square AFGH. H shall be the point required, that is, the rectangle AB, BH shall equal the square on AH. Or completing the parallelogram CFGL the proof is similar to that of II. 11.



In the figure of II. 11 prove that

**2.** If CH be produced to meet BF at L, CL is at right angles to BF.

[The  $\triangle^s$  HCA, FBA are equal in all respects.  $\therefore$  etc.]

**3.** If BE and CH meet at O, AO is at right angles to CH.

[ $\angle$  EFB =  $\angle$  EBF.  $\therefore$ , by Ex. 2,  $\angle$  ECO =  $\angle$  HOB =  $\angle$  EOC;  $\therefore$  EO = EC = EA, etc.]

**4.** GB, FD, and AK are all parallel.

[ $2\triangle$  GFB = FH = HD =  $2\triangle$  DGB;  $\therefore$  etc.,

$2\triangle$  GAB =  $2\triangle$  GHB + FH =  $2\triangle$  GHB + HD

=  $2\triangle$  GHB +  $2\triangle$  HKB =  $2\triangle$  GBK;  $\therefore$  etc.]

**5.** If FG meet DB in M, then CHM is a straight line.

[Use the converse of I. 43.]

**6.** The rectangles GD, FB, AK are all equal.

**7.** KF and HD are parallel.

[ $2\triangle$  KHF = AK = DG [by Ex. 6] =  $\triangle$  KDF;  $\therefore$  etc.]

**8.** The rectangle AH, HB is equal to the difference of the squares on AH, HB.

**9.**  $AB^2 + BH^2 = 3AH^2$ .

**10.** The square on the sum of AB, BH = five times the square on AH.

**11.** CF is divided at A in medial section.

**12.** The square on EF = five times the square on EA.

**13.** If in HA a point P be taken so that HP = HB, then AH is divided in medial section at P.

**14.** Shew that in a straight line, divided as in II. 11, the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

**15.** Produce a given straight line so that the rectangle contained by the whole straight line thus produced and the part produced may be equal to the square on the given straight line.

[In the figure of II. 11, if CA be the given straight line, then F is the required point; hence the required construction.]

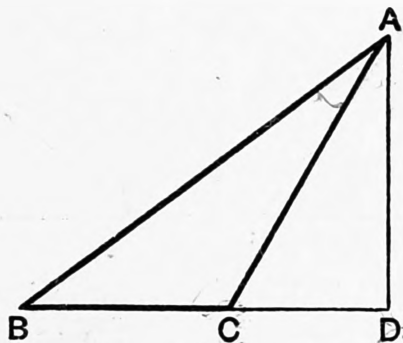
*Note.* In the figures of the two following propositions the line BD is often called the **projection** of the side AB upon the base BC.

## PROPOSITION 12. THEOREM.

*In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.*

Let  $ABC$  be an obtuse-angled triangle having the obtuse angle  $ACB$ , and from the point  $A$  let  $AD$  be drawn perpendicular to  $BC$  produced:

*the square on  $AB$  shall be greater than the squares on  $AC$ ,  $CB$ , by twice the rectangle  $BC$ ,  $CD$ .*



**Proof.** Because  $BD$  is divided at  $C$ , the sq. on  $BD$  = sqs. on  $BC$ ,  $CD$ , and twice the rect.  $BC$ ,  $CD$ . [II. 4.]  
To each add the square on  $DA$ ;

$\therefore$  the squares on  $BD$ ,  $DA$  = the squares on  $BC$ ,  $CD$ ,  $DA$ ,  
and twice the rectangle  $BC$ ,  $CD$ . [Axiom 2.]

But the square on  $BA$  = the squares on  $BD$ ,  $DA$ , } [I. 47.]  
and the square on  $CA$  = the squares on  $CD$ ,  $DA$ , }  
because the angle  $D$  is a right angle;

$\therefore$  the square on  $BA$  = the squares on  $BC$ ,  $CA$ , and twice the rectangle  $BC$ ,  $CD$ ;

that is, the square on  $BA$  is greater than the squares on  $BC$ ,  $CA$  by twice the rectangle  $BC$ ,  $CD$ .

Wherefore, in obtuse-angled triangles, etc.

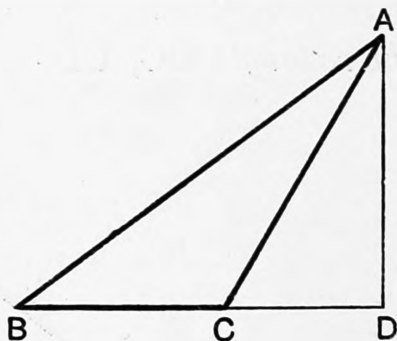
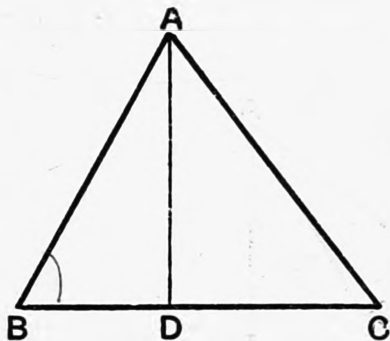
[Q.E.D.]

## PROPOSITION 13. THEOREM.

*In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.*

Let  $ABC$  be any triangle, and the angle at  $B$  an acute angle; on  $BC$  let fall the perpendicular  $AD$  from the opposite angle:

*the square on  $AC$  shall be less than the squares on  $CB$ ,  $BA$ , by twice the rectangle  $CB$ ,  $BD$ .*



**Proof.** First, let  $AC$  be not perpendicular to  $BC$ .

In Fig. 1  $BC$  is divided at  $D$ , and in Fig. 2  $BD$  is divided at  $C$ ; therefore, in both cases, the squares on  $CB$ ,  $BD$  = twice the rectangle  $CB$ ,  $BD$ , together with the square on  $CD$ . [II. 7.]

To each add the square on  $DA$ ;

$\therefore$  the squares on  $CB$ ,  $BD$ ,  $DA$  = twice the rectangle  $CB$ ,  $BD$ , together with the squares on  $CD$ ,  $DA$ .

But the square on  $BA$  = the squares on  $BD$ ,  $DA$ ,

and the square on  $CA$  = the squares on  $CD$ ,  $DA$ ; [I. 47.]

$\therefore$  the squares on  $CB$ ,  $BA$  = twice the rectangle  $CB$ ,  $BD$ , together with the square on  $AC$ .

that is,

the square on  $AC$  alone is less than the squares on  $AB$ ,  $BC$  by twice the rectangle  $CB$ ,  $BD$ .

Secondly, let AC be perpendicular to BC.



Then BC is the straight line between the perpendicular and the acute angle at B ;  
and it is clear that the squares on AB, BC  
= the sq. on AC, and twice the square on BC. [I. 47 and Ax. 2.  
Wherefore, *in every triangle, etc.* [Q.E.D.

### EXERCISES ON PROPOSITIONS 12 AND 13.

**\*\*1.** *In any triangle the sum of the squares on the sides is equal to twice the square on half the base, together with twice the square on the line joining the vertex to the middle point of the base (that is, together with twice the square on the median through the vertex).*

Let D be the middle point of the base BC of the triangle ABC.

If ADB is a right angle the theorem is clear by I. 47. If not, of the two angles ADB, ADC, one is obtuse and the other acute.

Let ADB be the obtuse angle.

Draw AE perpendicular to BC.

Then by Euc. II. 12, 13,

square on AB = squares on AD, DB + twice rect. BD, DE.

and square on AC = squares on AD, DC - twice rect. DC, DE

= squares on AD, DB - twice rect. BD, DE,

since BD and DC are equal ;

∴ by addition,

the squares on AB, AC = twice the squares on AD, DB.

The student will have no difficulty in drawing the figure (or see App., Art. 32). There will be two cases according as the angle ACB is acute or obtuse.

The above is a very important proposition. Many of the following exercises depend on it.

**\*\*2.** ABCD is a rectangle and P any point ; prove that the squares on PA, PC are together equal to the squares on PB, PD.

**\*\*3.** Four times the sum of the squares on the medians of a triangle is equal to three times the sum of the squares on the sides of the triangle.

**\*\*4.** The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

**5.** The base of a triangle is given and is bisected by the centre of a given circle ; if the vertex be at any point of the circumference, shew that the sum of the squares on the two sides of the triangle is invariable.

**6.** In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [Use Ex. 4, and Page 72, Ex. 2.]

**7.** If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares on the straight lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

**8.** The sides of a triangle are 8, 12, and 13 inches ; prove that it is acute-angled.

**9.** The sides of a triangle are 8, 12, and 15 inches ; prove that it is obtuse-angled.

**10.** If the angle ACB in II. 12 become more and more obtuse, until finally A falls on BC produced, what does the proposition become ?

**11.** The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and by the straight line intercepted between the perpendicular let fall on the side from the opposite angle and the extremity of the base.

**12.** If one angle of a triangle be equal to two-thirds of a right angle the square on the opposite side is less than the sum of the squares containing that angle by the rectangle contained by these two sides.

**13.** Find the obtuse angle of a triangle, when the square on the side opposite to the obtuse angle is greater than the sum of the squares on the sides containing it by the rectangle of the sides.

**14.** ABC is a triangle having the sides AB and AC equal ; if AB is produced beyond the base to D so that BD is equal to AB, shew that the square on CD is equal to the square on AB, together with twice the square on BC.

**15.** In AB the diameter of a circle take two points C and D equally distant from the centre, and from any point E in the circumference draw EC, ED ; shew that the squares on EC and ED are together equal to the squares on AC and AD. [Use Ex. 1 and II. 10.]

**16.** In BC the base of a triangle take D such that the squares on AB and BD are together equal to the squares on AC and CD ; then the middle point of AD will be equally distant from B and C.

**17.** A square BDEC is described on the hypotenuse BC of a right-angled triangle ABC: shew that the squares on DA and AC are together equal to the squares on EA and AB. [Use Ex. 1.]

**18.** ABC is an acute-angled triangle, and BE, CF are the perpendiculars on CA, AB; prove that the rectangles AB, AF and AC, AE are equal.

[By II. 13,  $AB^2 + AC^2 - 2BA \cdot AF = BC^2 = AB^2 + AC^2 - 2AC \cdot AE$ ;  $\therefore$  etc.]

**19.** In a triangle ABC the angles B and C are acute: if E and F be the points where perpendiculars from the opposite angles meet the sides AC, AB, shew that the square on BC is equal to the rectangle AB, BF, together with the rectangle AC, CE.

**20.** Describe an isosceles obtuse-angled triangle such that the square on the largest side may be equal to three times the square on either of the equal sides. [The obtuse  $\angle$  must = four-thirds of a right  $\angle$ .]

**21.** ABC is an equilateral triangle, and AB is produced to D so that BD is twice AB. Prove that the square on CD is seven times the square on AB.

**22.** Squares are described on the sides of any triangle; the sum of the squares on the straight lines joining adjacent corners of the squares are equal to three times the sum of the squares on the sides.

[The letters being as in I. 47, produce BC to X, so that  $BC = CX$ . The  $\triangle$  KCE, ACX are equal in all respects, so that  $KE = AX$ ;

$$\therefore KE^2 + AB^2 = AX^2 + AB^2 = 2AC^2 + 2BC^2, \text{ by Ex. 1.}$$

Similarly for  $HG^2, FD^2$ ;  $\therefore$  etc.]

**23.** The squares on the two equal sides of an isosceles triangle are together less than the squares on the two sides of any other triangle on the same base and between the same parallels. [Use Ex. 1.]

**\*\*24.** The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

[Let E, F be the middle points of the diagonals AC, BD of the quad<sup>l</sup> ABCD. Then, by Ex. 1

$$AB^2 + BC^2 = 2BE^2 + 2EC^2, \text{ and } AD^2 + DC^2 = 2DE^2 + 2EC^2;$$

$\therefore$  sum of the squares on the sides

$$= 4EC^2 + 2BE^2 + 2DE^2 = 4EC^2 + 4EF^2 + 4DF^2 = AC^2 + BD^2 + 4EF^2.]$$

**25.** If the squares on the sides of a quadrilateral are together equal to the squares on its diagonals it must be a parallelogram.

**26.** Construct a triangle having given its area, its base, and the sum of the squares on its sides.

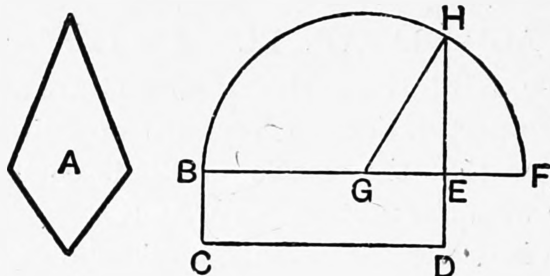
[By I. 39 the vertex lies on a line parallel to the base, and by Ex. 1 its distance from the middle point of the base is given.]

## PROPOSITION 14. PROBLEM.

*To describe a square that shall be equal to a given rectilineal figure.*

Let *A* be the given rectilineal figure :

*it is required to describe a square that shall be equal to A.*



**Construction.** Describe the rectangular parallelogram *BCDE* equal to the figure *A*. [I. 45.]

Then, if its sides *BE*, *ED* are equal, it is a square, and what was required is now done.

But if they are not equal, produce one of them *BE* to *F*,

make *EF* equal to *ED*, [I. 3.]

and bisect *BF* at *G*; [I. 10.]

with centre *G*, and radius *GB*, or *GF*, describe the semi-circle *BHF*, and produce *DE* to meet it in *H*.

The square on *EH* shall = the figure *A*. Join *GH*.

**Proof.** Because *BF* is divided into two equal parts at *G*, and into two unequal parts at *E*,

the rectangle *BE*, *EF*, together with the square on *GE*

= the square on *GF*, [II. 5.]

that is, = the square on *GH*, since *GF* = *GH*, [Constr.]

that is, = the squares on *GE*, *EH*; [I. 47.]

Take away the square on *GE*, which is common to both;

∴ the rectangle *BE*, *EF* = the square on *EH*. [Axiom 3.]

But *BD* is the rectangle *BE*, *EF*, since *EF* = *ED*; [Constr.]

∴ *BD* = the square on *EH*.

But *BD* = the rectilineal figure *A*; [Construction.]

∴ the square on *EH* = the rectilineal figure *A*.

Wherefore a square has been made equal to the given rectilineal figure *A*, namely, the square described on *EH*. [Q. E. F.]



## BOOK III.

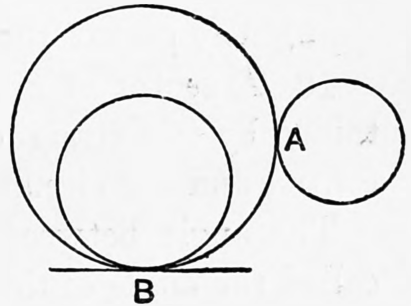
### DEFINITIONS.

1. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

[This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.]

2. A straight line is said to **touch** a circle when it meets the circle, and, being produced, does not cut it.

Such a straight line is called a **tangent**, and the point at which it touches the circle is called its **point of contact**.



3. Circles are said to **touch** one another which meet but do not cut one another.

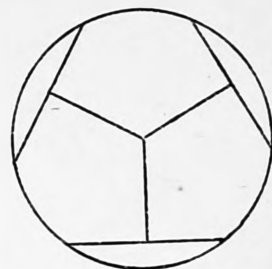
When each circle is without the other, as at A, they are said to touch **externally**; when one is within the other, as at B, they are said to touch **internally**.

4. A **secant** of a circle is a straight line which cuts the circumference in two points.

5. A **chord** of a circle is a straight line which joins any two points on the circle.



6. Chords are said to be **equally distant** from the centre of a circle when the perpendiculars drawn to them from the centre are equal.



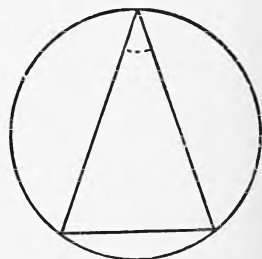
Also the chord on which the greater perpendicular falls is said to be farther from the centre.

7. A **segment** of a circle is the figure contained by a chord and the circumference it cuts off.



The chord of the segment is sometimes called its **base**.

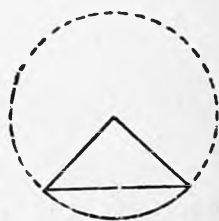
8. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.



Also an angle is said to insist or stand on the circumference intercepted between the straight lines which contain the angle.

9. Any part of the circumference of a circle is called an **arc**.

10. A **sector** of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



The angle between the straight lines is called the angle of the sector.

11. Similar segments of circles are those in which the angles are equal, or which contain equal angles.



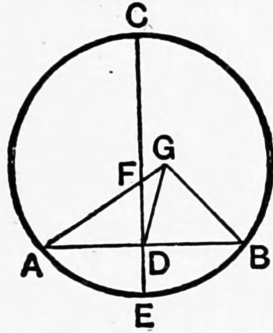
12. Two circles are said to be **concentric** when they have the same centre.

[*Note.* In the following propositions, whenever the expression “straight lines from the centre,” or “drawn from the centre,” occurs, it is to be understood that the lines are drawn to the circumference.]

## PROPOSITION 1. PROBLEM.

*To find the centre of a given circle.*

Let  $ABC$  be the given circle :  
*it is required to find its centre.*



**Construction.** Draw any chord  $AB$ , and bisect it at  $D$ ; [I. 10.  
 from  $D$  draw  $DC$  at right angles to  $AB$ ; [I. 11.  
 produce  $CD$  to meet the circumference at  $E$ , and bisect  $CE$   
 at  $F$ . [I. 10.

Then  $F$  shall be the centre of the circle  $ABC$ .

For if  $F$  be not the centre, if possible, let  $G$  be the centre;  
 and join  $GA$ ,  $GD$ ,  $GB$ .

**Proof.** In the triangles  $GDA$ ,  $GDB$ ,  
 because  $\begin{cases} DA = DB, \\ \text{and } DG \text{ is common to both,} \\ \text{and } GA = GB, \end{cases}$  [Construction.  
 because they are drawn from the centre  $G$ ; [I. Definition 15.  
 $\therefore$  the  $\angle ADG =$  the  $\angle BDG$ , [I. 8.  
 these angles, being adjacent, are right angles; [I. Def. 10.  
 $\therefore$  the  $\angle BDG$  is a right  $\angle$ .  
 But the  $\angle BDF$  is also a right  $\angle$ ; [Construction.  
 $\therefore$  the  $\angle BDG =$  the  $\angle BDF$ , [Axiom 11.  
 the less to the greater; which is impossible;  
 $\therefore$   $G$  is not the centre of the circle  $ABC$ .

In the same manner it may be shewn that no other point  
 out of the line  $CE$  is the centre;

and since  $CE$  is bisected at  $F$ , any other point in  $CE$  divides it into unequal parts, and cannot be the centre.

$\therefore$  no point but  $F$  is the centre ;

that is,  $F$  is the centre of the circle  $ABC$  :

*which was to be found.*

**Corollary.** If in a circle a straight line bisect another at right angles, the centre of the circle is in the straight line which bisects the other.

### EXERCISES.

**\*\*1.** The line joining the centres of two circles which meet in two points is perpendicular to the line joining the two points, and bisects it.

**2.** Shew that the straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

**3.** Find the shortest distance between two circles which do not meet.

**4.** Given two chords of a circle in magnitude and position, find its centre.

**5.** Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

**6.** Describe a circle which shall have its centre in a given straight line and pass through two given points.

**7.** Describe a circle which shall pass through two given points and have a given radius.

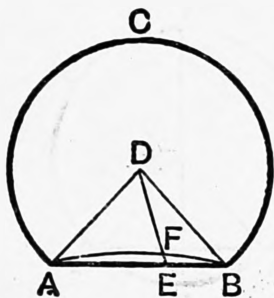
## PROPOSITION 2. THEOREM.

*If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.*

Let ABC be a circle, and A and B any two points in the circumference :

*the straight line drawn from A to B shall fall within the circle.*

For if it do not, let it fall, if possible, without, as AEB.



**Construction.** Find D, the centre of the circle ABC; [III. 1. join DA, BD; in the arc AB take any point F, join DF, and produce it to meet AB at E.

**Proof.** Because  $DA = DB$ , [I. Definition 15.  
the  $\angle DAB =$  the  $\angle DBA$ . [I. 5.

But the exterior  $\angle DEB$  of the  $\triangle DAE$  is greater than the interior opposite  $\angle DAE$ , [I. 16.

and the  $\angle DAE$  was shewn to be equal to the  $\angle DBE$ ;

$\therefore$  the  $\angle DEB$  is greater than the  $\angle DBE$ ;

$\therefore$  DB is greater than DE. [I. 19.

But  $DB = DF$ ; [I. Definition 15.

$\therefore$  DF is greater than DE, the less than the greater; which is impossible;

$\therefore$  the straight line AB does not fall without the circle.

In the same manner it may be shewn that it does not fall on the circumference.

Therefore it falls within the circle.

Wherefore, *if any two points, etc.*

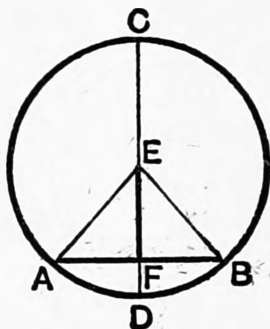
[Q.E.D.

## PROPOSITION 3. THEOREM.

*If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and if it cut it at right angles, it shall bisect it.*

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, at the point F:

*CD shall cut AB at right angles.*



**Construction.** Take E, the centre of the circle; and join EA, EB. [III. 1.]

**Proof.** In the triangles AFE, BFE,  
 because  $\left\{ \begin{array}{l} AF = FB, \\ \text{and } FE \text{ is common,} \\ \text{and the base } EA = \text{the base } EB; \end{array} \right.$  [Hypothesis.]  
 $\therefore$  the  $\angle AFE = \text{the } \angle BFE$ ; [I. Definition 15.]  
 $\therefore$  these angles being adjacent, are right angles. [I. 8.]  
 $\therefore$  CD cuts AB at right angles. [I. Def. 10.]

*Conversely:* Let CD cut AB at right angles;  
 CD shall also bisect AB, that is, AF shall be equal to FB.

The same construction being made, because  $EA = EB$ ,  
 the  $\angle EAF = \text{the } \angle EBF$ . [I. 5.]

Then in the triangles AFE, BFE,  
 because  $\left\{ \begin{array}{l} \text{the } \angle EAF = \text{the } \angle EBF, \\ \text{and the right } \angle AFE = \text{the right } \angle BFE, \\ \text{and the side } EF \text{ is common;} \end{array} \right.$   
 $\therefore AF = FB$ . [I. 26.]

Wherefore, *if a straight line, etc.*

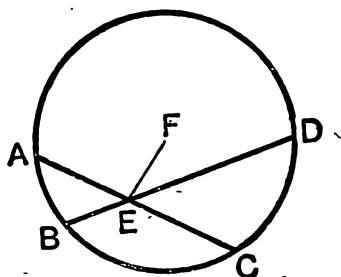
[Q.E.D.]

## PROPOSITION 4. THEOREM.

*If in a circle chords cut one another which do not both pass through the centre, they do not bisect one another.*

Let ABCD be a circle, and AC, BD two chords in it, which cut one another at the point E, and do not both pass through the centre :

*then AC, BD shall not bisect one another.*



CASE I. If one of the chords pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre.

CASE II. If neither of them pass through the centre, if possible, let  $AE = EC$  and  $BE = ED$ .

**Construction.** Find F, the centre of the circle, [III. 1.  
and join EF.

**Proof.** Because FE, passing through the centre, bisects AC which does not pass through the centre ; [Hypothesis.

$\therefore$  the  $\angle FEA$  is a right  $\angle$ . [III. 3.

Again, because FE bisects BD, [Hypothesis.

$\therefore$  the  $\angle FEB$  is a right  $\angle$ . [III. 3.

But the  $\angle FEA$  was shewn to be a right  $\angle$  ;

$\therefore$  the  $\angle FEA =$  the  $\angle FEB$ , [Axiom 11.

the less to the greater ; which is impossible.

$\therefore$  AC, BD do not bisect each other.

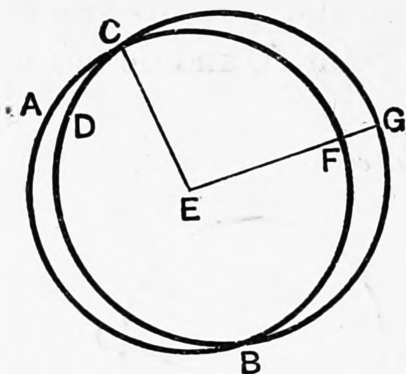
Wherefore, *if in a circle, etc.* [Q.E.D.

## PROPOSITION 5. THEOREM.

*If two circles cut one another, they shall not have the same centre.*

Let the two circles ABC, CDG cut one another at the points B, C:

*they shall not have the same centre.*



**Construction.** If it be possible, let E be their centre ; join EC, and draw any straight line EFG, meeting the circumferences at F and G.

**Proof.** Because E is the centre of the circle ABC,

$$\therefore EC = EF. \quad [\text{I. Definition 15.}]$$

Again, because E is the centre of the circle CDG,

$$\therefore EC = EG. \quad [\text{I. Definition 15.}]$$

$$\therefore EF = EG, \quad [\text{Axiom 1.}]$$

the less to the greater, which is impossible ;

$\therefore$  E is not the centre of the circles ABC, CDG.

Wherefore, *if two circles, etc.*

[Q.E.D.]

## EXERCISES ON PROPOSITIONS 3 AND 4.

**\*\*1.** Through a given point within a given circle draw the chord which is bisected at the point.

**\*\*2.** The middle points of all parallel chords of a circle lie on a straight line which passes through the centre and is perpendicular to them.

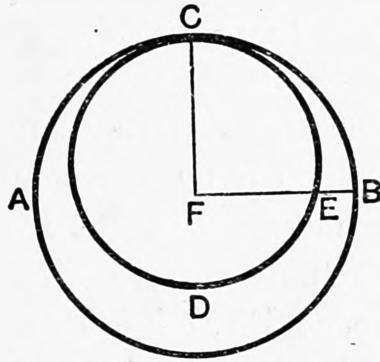
**\*\*3.** The middle points of all chords of a circle, which are of the same length, all lie on a concentric circle.

**4.** If two circles ABCD, ABEF cut each other in points A, B, any two parallel straight lines DAF, CBE drawn through the points of section to cut the circles are equal. [Use Ex. 2.]

## PROPOSITION 6. THEOREM.

*If two circles touch one another internally, they shall not have the same centre.*

Let the two circles ABC, CDE touch one another internally at the point C:  
*they shall not have the same centre.*



**Construction.** If it be possible, let F be their centre; join FC, and draw any straight line FEB, meeting the circumferences at E and B.

**Proof.** Because F is the centre of the circle ABC,

$$FC = FB. \quad [\text{I. Definition 15.}]$$

Again, because F is the centre of the circle CDE,

$$FC = FE; \quad [\text{I. Definition 15.}]$$

$$\therefore FE = FB, \quad \text{as. 1}$$

the less to the greater, which is impossible;

$\therefore$  F is not the centre of the circles ABC, CDE.

Wherefore, *if two circles, etc.*

[Q.E.D.]

**Note.** Propositions 5 and 6 amount to this proposition: *If two circles meet in a point they cannot have the same centre; so that circles which have the same centre and one point common must coincide altogether.*



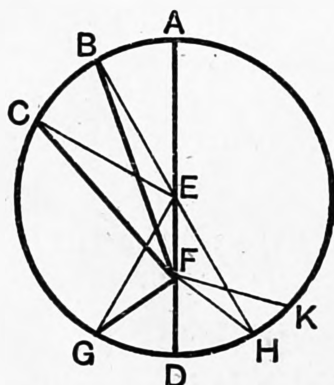
## PROPOSITION 7. THEOREM.

*If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from this point to the circumference, then*

- (1) *the greatest is that in which the centre is, and*
- (2) *the other part of the diameter is the least ;*
- (3) *of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote ; and*
- (4) *from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let ABCD be a circle and AD its diameter, in which let any point F be taken which is not the centre ; let E be the centre : of all the straight lines FB, FC, FG, etc., that can be drawn from F to the circumference,

- (1) *FA, which passes through E, shall be the greatest, and*
- (2) *FD, the other part of the diameter AD, shall be the least ; and*
- (3) *of the others FB shall be greater than FC, and FC than FG.*



**Construction.** Join BE, CE, GE.

**Proof.** (1) Because any two sides of a triangle are together greater than the third side, [I. 20.]

therefore BE, EF are together greater than BF.

But  $BE = AE$  ; [I. Definition 15.]

$\therefore$  AE, EF are together greater than BF,

that is, AF is greater than BF.

Similarly it can be shewn that FA is greater than any other line drawn from F to the circumference ;

$\therefore$  FA is the greatest of all such lines.

(2) Because GF, FE are together greater than EG, [I. 20.  
and that EG = ED; [I. Definition 15.

∴ GF, FE are together greater than ED.

Take away the common part FE, and the remainder GF is greater than the remainder FD.

Similarly any other straight line drawn from F can be shewn to be greater than FD;

∴ FD is the least of all such lines.

(3) In the triangles BEF, CEF,

because  $\begin{cases} BE = CE, \\ \text{and } EF \text{ is common,} \\ \text{but the } \angle BEF \text{ is greater than the } \angle CEF; \end{cases}$  [I. Definition 15.

∴ the base BF is greater than the base CF. [I. 24.

Similarly it may be shewn that CF is greater than GF.

(4) Also, there can be drawn two equal straight lines from the point F to the circumference, one on each side of the shortest line FD.

At the point E make the  $\angle FEH$  equal to the  $\angle FEG$ , [I. 23.  
and join FH.

Then in the triangles GEF, HEF,

because  $\begin{cases} EG = EH, \\ \text{and } EF \text{ is common,} \\ \text{and the } \angle GEF = \text{the } \angle HEF; \end{cases}$  [I. Definition 15.  
[Construction.  
∴ the base FG = the base FH. [I. 4.

But, besides FH, no other straight line can be drawn from F to the circumference equal to FG.

For, if it be possible, let FK be equal to FG.

Then, because FK = FG, and FH = FG; [Hypothesis.  
∴ FH = FK; [Axiom 1.

that is, a line nearer to that which passes through the centre is equal to a line which is more remote,  
which is impossible by what has been already shewn.

Wherefore, if any point be taken, etc.

III. 7.  
[Q.E.D.

## PROPOSITION 8. THEOREM.

*If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre ;*

*(1) of those which fall on the concave circumference, the greatest is that which passes through the centre ; and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote ; but*

*(2) of those which fall on the convex circumference, the least is that between the point without the circle and the diameter ; and of the rest, that which is nearer to the least is always less than one more remote ; and*

*(3) from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let  $ABC$  be a circle, and  $D$  any point without it, and from  $D$  let the straight lines  $DA$ ,  $DE$ ,  $DF$ ,  $DC$  be drawn to the circumference, of which  $DA$  passes through the centre ;

*(1) of those which fall on the concave circumference  $AEFC$ , the greatest shall be  $DA$ , and  $DE$  shall be greater than  $DF$ , and  $DF$  than  $DC$  ; but*

*(2) of those which fall on the convex circumference  $GKLH$ , the least shall be  $DG$ , and  $DK$  less than  $DL$ , and  $DL$  than  $DH$ .*

**Construction.** Take  $M$ , the centre of the circle  $ABC$ , [III. 1. and join  $ME$ ,  $MF$ ,  $MC$ ,  $MH$ ,  $ML$ ,  $MK$ .

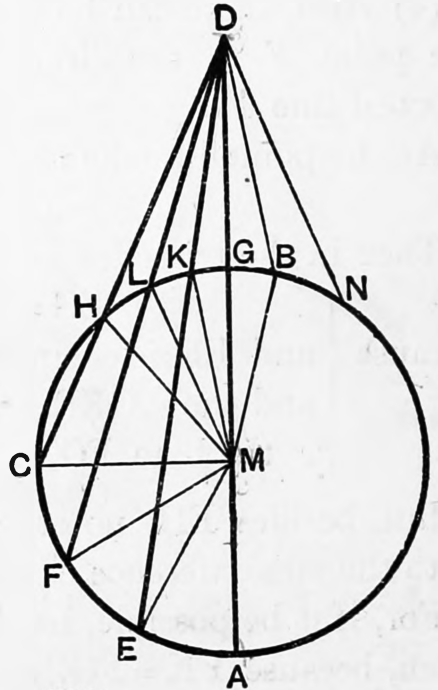
**Proof.** (1) In the  $\triangle EMD$  the two sides  $EM$ ,  $MD$  are together greater than  $ED$ . [I. 20.

But  $EM = AM$  ; [I. Definition 15.

$\therefore AM$ ,  $MD$  are together greater than  $ED$ ,  
that is,  $AD$  is greater than  $ED$ .

Again, in the triangles  $EMD$ ,  $FMD$ ,

because  $\begin{cases} EM = FM, \\ \text{and } MD \text{ is common,} \\ \text{but the } \angle EMD \text{ is greater than the } \angle FMD; \end{cases}$



$\therefore$  the base ED is greater than the base FD. [I. 24.]

Similarly, it may be shewn that FD is greater than CD.  
Therefore DA is the greatest, and DE greater than DF, and DF greater than DC.

(2) Again, because MK, KD are greater than MD, [I. 20.]  
and  $MK = MG$ , [I. Definition 15.]  
the remainder KD is greater than the remainder GD,  
that is, GD is less than KD.

Also, in the  $\triangle MLD$ , because straight lines MK, KD are drawn to a point K within the triangle from the ends of its base ;  
 $\therefore$  MK, KD are less than ML, LD ; [I. 21.]

but  $MK = ML$  ; [I. Definition 15.]  
 $\therefore$  the remainder KD is less than the remainder LD.

Similarly, it may be shewn that LD is less than HD ;  
 $\therefore$  DG is the least, and DK less than DL, and DL than DH.

(3) Also, there can be drawn two equal straight lines from D to the circumference, one on each side of the least line.

**Construction.** At the point M, in the straight line MD, make the angle DMB equal to the angle DMK, [I. 23.]  
and join DB.

**Proof.** In the triangles KMD, BMD,  
because  $\left\{ \begin{array}{l} MK = MB, \\ \text{and MD is common,} \\ \text{and the } \angle DMK = \text{the } \angle DMB ; \end{array} \right.$  [I. Def. 15.]  
[Constr.]  
 $\therefore$  the base DK = the base DB. [I. 4.]

But, besides DB, no other straight line can be drawn from D to the circumference equal to DK.

For, if it be possible, let  $DN = DK$ .  
Then, because  $DB = DK$ ,  $\therefore DB = DN$  ; [Axiom 1.]  
that is, a line nearer to the least = one which is more remote,  
which is impossible by what has been already shewn.

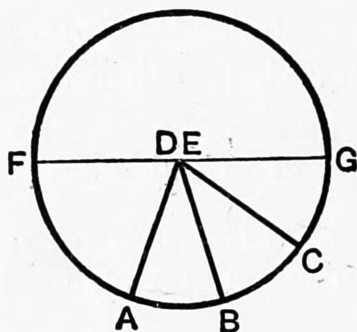
Wherefore, if any point be taken, etc. [Q.E.D.]

## PROPOSITION 9. THEOREM.

*If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference, that point is the centre of the circle.*

Let the point  $D$  be taken within the circle  $ABC$ , from which, to the circumference, there are drawn more than two equal straight lines, namely,  $DA$ ,  $DB$ ,  $DC$ :  
*the point  $D$  shall be the centre of the circle.*

For, if not, let  $E$  be the centre.



**Construction.** Join  $DE$ , and produce it both ways to meet the circumference at  $F$  and  $G$ .

**Proof.**  $D$  is a point within the circle which is not its centre, and from it more than two equal straight lines, namely, the three straight lines  $DA$ ,  $DB$ ,  $DC$ , have been drawn to the circumference;

but this is impossible;

[III. 7.]

$\therefore E$  is not the centre of the circle  $ABC$ .

In the same manner it may be shewn that any other point than  $D$  is not the centre;

therefore  $D$  is the centre of the circle  $ABC$ .

Wherefore, *if a point be taken, etc.*

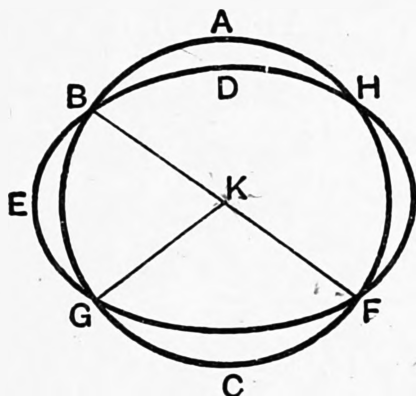
[Q. E. D.]

*Note.* For other demonstrations of Propositions 9 and 10 see Notes, Pages 325 and 326.

## PROPOSITION 10. THEOREM.

*One circumference of a circle cannot cut another at more than two points.*

If it be possible, let the circumference ABC cut the circumference DEF at more than two points, namely, at the points B, G, F.



**Construction.** Take K, the centre of the circle ABC,  
and join KB, KG, KF. [III. 1.]

**Proof.** Because K is the centre of the circle ABC,  
∴ KB, KG, KF are all equal. [I. Definition 15.]

And because within the circle DEF, the point K is taken,  
from which, to the circumference DEF, are drawn more than  
two equal straight lines KB, KG, KF,

∴ K is the centre of the circle DEF. [III. 9.]

But K is also the centre of the circle ABC; [Construction.]  
∴ the same point is the centre of two circles which cut one  
another, which is impossible. [III. 5.]

Wherefore, *one circumference, etc.* [Q.E.D.]

## EXERCISES.

1. Two circles cannot meet in three points without coinciding entirely.

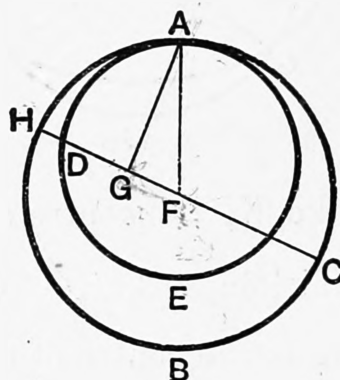
2. Through three points, which are not in the same straight line, only one circle can be drawn.

## PROPOSITION 11. THEOREM.

*If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.*

Let the two circles  $ABC$ ,  $ADE$  touch one another internally at the point  $A$ ; let  $F$  be the centre of the circle  $ABC$ , and  $G$  the centre of the circle  $ADE$ :

*the straight line  $FG$ , being produced, shall pass through the point  $A$ .*



**Proof.** For, if not, let it pass otherwise, if possible, as  $FGDH$ , and join  $AF$ ,  $AG$ .

Then, because  $AG$ ,  $GF$  are together greater than  $AF$ , [I. 20.  
and  $AF = HF$ ; [I. Definition 15.

$\therefore AG$ ,  $GF$  are together greater than  $HF$ .

Take away the common part  $GF$ ;

$\therefore$  the remainder  $AG$  is greater than the remainder  $HG$ .

But  $AG = DG$ ; [I. Definition 15.

$\therefore DG$  is greater than  $HG$ , the less than the greater,  
which is impossible.

$\therefore FG$ , being produced, cannot pass otherwise than through the point  $A$ ,

that is, it must pass through  $A$ .

Wherefore, *if two circles, etc.*

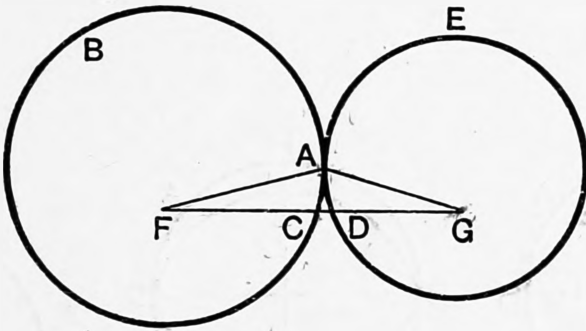
[Q. E. D.]

## PROPOSITION 12. THEOREM.

*If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.*

Let the two circles ABC, ADE touch one another externally at the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE:

*the straight line FG shall pass through the point A.*



**Proof.** For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG.

Then, because F is the centre of the circle ABC,

$$FA = FC; \quad [\text{I. Definition 15.}]$$

and because G is the centre of the circle ADE,  $GA = GD$ ;

$$\therefore FA, AG \text{ are equal to } FC, DG; \quad [\text{Axiom 2.}]$$

$\therefore$  the whole FG is greater than FA, AG.

But FG is also less than FA, AG, [I. 20.]

which is impossible;

$\therefore$  the straight line FG cannot pass otherwise than through the point A,

that is, it must pass through A.

Wherefore, *if two circles, etc.*

[Q.E.D.]

**Note.** Propositions 11 and 12 may be enunciated in one, thus: *If two circles touch one another, their two centres and the point of contact are in a straight line.*

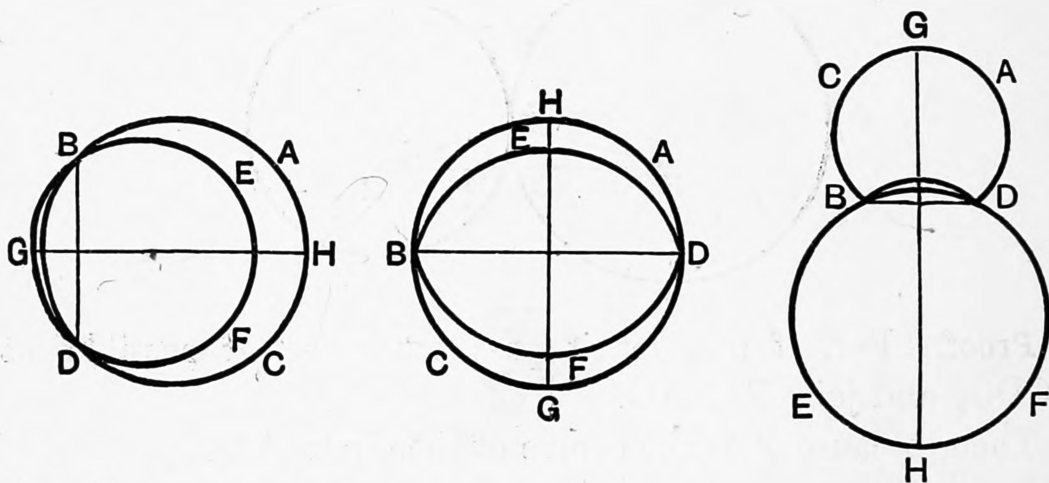


## PROPOSITION 13. THEOREM.

*One circle cannot touch another at more points than one, whether it touches it on the inside or outside.*

For, if it be possible, let the circle EBF touch the circle ABC at more points than one, namely, at B and D, internally as in Figs. 1 and 2, or externally as in Fig. 3.

Join BD, and draw GH bisecting BD at right angles. [I. 10, 11.



**Proof.** Because, in all three figures, BD is a chord of each circle, and GH bisects it at right angles ;

$\therefore$  the centre of each circle is in GH.

[III. 1, *Corollary*.

$\therefore$  GH passes through the point of contact.

[III. 11, 12.

But GH does not pass through the point of contact, because the points B, D are out of the line GH,

which is absurd.

Wherefore, *one circle cannot, etc.*

[Q.E.D.

**EXERCISES.**

**\*\*1.** If the distance between the centres of two circles be equal to the sum of their radii, the circles touch externally.

**\*\*2.** If it be equal to their difference, the circles touch internally.

**3.** Describe a circle with a given centre which shall touch a given circle. How many such circles can be drawn?

**4.** Describe a circle to pass through a given point and touch a given circle at a given point.

[Let  $B$  be the given point on the given circle whose centre is  $O$  and  $A$  the other given point. Join  $AB$ ; produce  $OB$  to  $C$ , and at  $A$  make the  $\angle BAD$  equal to the  $\angle ABC$ ; let  $AD$  meet  $OC$  in  $D$ ; then  $D$  is the centre of the required circle, etc.]

**5.** Describe a circle with a given radius to pass through a given point and touch a given circle.

[The centre of the required circle is at a distance from the centre of the given circle equal to the sum of the given radius and the radius of the given circle; it is also at a distance from the given point equal to the given radius;  $\therefore$  etc.]

**6.** Describe a circle with a given radius to touch two given circles.

**7.** Three circles touch one another externally at the points  $A, B, C$ ; from  $A$  the straight lines  $AB, AC$  are produced to cut the circle  $BC$  at  $D$  and  $E$ . Show that  $DE$  is a diameter of  $BC$ , and is parallel to the straight line joining the centres of the other circles.

**8.** If in any two given circles which touch one another there be drawn two parallel diameters, an extremity of each diameter and the point of contact shall lie in the same straight line.

**9.** If circles be described on the two sides of a right-angled triangle as diameters, they will be touched by a circle whose centre is the middle point of the hypotenuse, and whose diameter is equal to the sum of the sides.

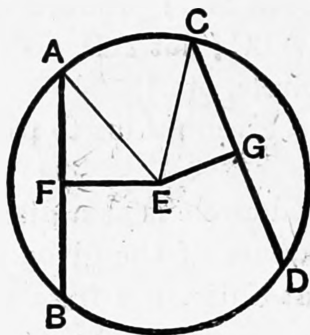
**10.** If a variable circle touch two fixed circles the sum, or the difference, of the distances of its centre from those of the fixed circles is equal to the sum, or the difference, of their radii.

## PROPOSITION 14. THEOREM.

*Equal chords in a circle are equally distant from the centre ; and those which are equally distant from the centre are equal to one another.*

Let the chords AB, CD in the circle ABCD be equal to one another :

*they shall be equally distant from the centre.*



**Construction.** Find E, the centre of the circle ABDC; [III. 1.  
from E draw EF, EG perpendiculars to AB, CD, [I. 12.  
and join EA, EC.

**Proof.** Because EF, passing through the centre, cuts AB, which does not pass through the centre, at right angles, it also bisects it; [III. 3.

$\therefore AF = FB$ , and AB is double of AF.

For the like reason CD is double of CG.

But  $AB = CD$ ;

[Hypothesis.

$\therefore AF = CG$ .

[Axiom 7.

Also because  $AE = CE$ ,

[I. Definition 15.

the square on AE = the square on CE.

But the square on AE = the squares on AF, FE, because the angle AFE is a right angle; [I. 47.

similarly the square on CE = the squares on CG, GE;

$\therefore$  the squares on AF, FE = the squares on CG, GE. [Axiom 1.

But the square on  $AF$  = the square on  $CG$ , since  $AF = CG$  ;  
 $\therefore$  the remaining sq. on  $FE$  = the remaining sq. on  $GE$  ; [Ax. 3.

$$\therefore EF = EG ;$$

$\therefore$   $AB, CD$  are equally distant from the centre. [III. Def. 6.

*Conversely* : Let  $AB, CD$  be chords equally distant from the centre, that is, let  $EF = EG$  ;

*then shall*  $AB = CD$ .

For, the same construction being made, it may be shewn, as before, that  $AB$  is double of  $AF$ , and  $CD$  double of  $CG$ , and that the squares on  $EF, FA$  = the squares on  $EG, GC$  ; but the square on  $EF$  = the square on  $EG$ , because  $EF = EG$  ;

[*Hypothesis*.

$\therefore$  the remaining sq. on  $FA$  = the remaining sq. on  $GC$  ; [Ax. 3.

$$\therefore AF = CG.$$

But  $AB$  was shewn to be double of  $AF$ , and  $CD$  double of  $CG$  ;

$$\therefore AB = CD.$$

[*Axiom* 6.

Wherefore, *equal straight lines, etc.*

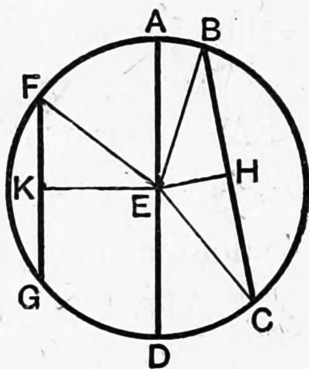
### EXERCISE.

In a given circle draw a chord of given length, not greater than the diameter, so that its centre may be on a given chord.

## PROPOSITION 15. THEOREM.

*The diameter is the greatest chord in a circle ;  
and, of all others, that which is nearer to the centre is always  
greater than one more remote ;  
and the greater is nearer to the centre than the less.*

Let ABCD be a circle, of which AD is a diameter, and E the centre ; and let BC be nearer to the centre than FG : AD shall be greater than any chord BC which is not a diameter, and BC shall be greater than FG.



**Construction.** From the centre E draw EH, EK perpendiculars to BC, FG, [I. 12.  
and join EB, EC, EF.

**Proof.** Because AE = BE, and ED = EC, [I. Definition 15.  
∴ AD = BE, EC ; [Axiom 2.  
but BE, EC are together greater than BC ; [I. 20.  
∴ also AD is greater than BC.

Also, because BC is nearer to the centre than FG, [Hypothesis.  
EH is less than EK. [III. Definition 6.

Now it may be shewn, as in the preceding proposition, that BC is double of BH, and FG double of FK, and that the squares on EH, HB = the squares on EK, KF.

But the square on EH is less than the square on EK,  
because EH is less than EK ;  
 $\therefore$  the square on HB is greater than the square on KF ;  
and therefore BH is greater than FK ;  
 $\therefore$  BC is greater than FG.

*Conversely* : Let BC be greater than FG : then, the same construction being made, EH shall be less than EK.

For, because BC is greater than FG, BH is greater than FK.  
But the squares on BH, HE = the squares on FK, KE ;  
and the square on BH is greater than the square on FK,  
because BH is greater than FK ;  
 $\therefore$  the square on HE is less than the square on KE,  
and therefore EH is less than EK.

Wherefore, *the diameter, etc.*

[Q. E. D.]

### EXERCISE.

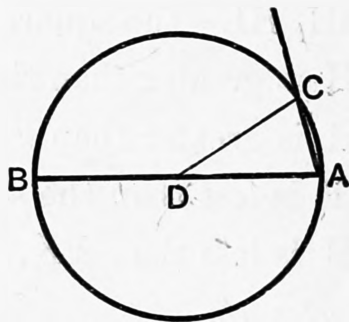
\*\*Through a given point within a circle, draw the shortest chord.

## PROPOSITION 16. THEOREM.

(1) *The straight line drawn at right angles to the diameter of a circle from the extremity of it falls without the circle ;*  
 and (2) *any other straight line drawn from the extremity must cut the circle.*

Let ABC be a circle, of which D is the centre and AB a diameter :

*the straight line drawn at right angles to AB, from its extremity A, shall fall without the circle.*



For, if not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference.

(1) **Proof.** Because  $DA = DC$ ,

[I. Definition 15.

the  $\angle DAC =$  the  $\angle DCA$ .

[I. 5.

But the  $\angle DAC$  is a right  $\angle$  ;

[Hypothesis.

$\therefore$  the  $\angle DCA$  is a right  $\angle$  ;

$\therefore$  the angles  $DAC, DCA =$  two right angles,

which is impossible.

[I. 17.

$\therefore$  the straight line drawn from A at right angles to AB does not fall within the circle.

And in the same manner it may be shewn that it does not fall on the circumference ;

$\therefore$  it must fall without the circle, as AE.

(2) If possible, let AF be another straight line drawn through A which does not cut the circle ; and from the centre D draw DG perpendicular to AF ;

[I. 12.

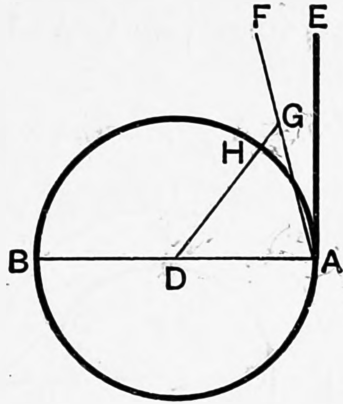
let DG meet the circumference at H.

Then, because the  $\angle DGA$  is a right  $\angle$ ,  
 the  $\angle DAG$  is less than a right  $\angle$  ;  
 $\therefore DA$  is greater than  $DG$ .

[Construction.

[I. 17.

[I. 19.



But  $DA = DH$  ; [I. Definition 15.

$\therefore DH$  is greater than  $DG$ , the less than the greater, which is impossible ;

$\therefore$  no straight line, except  $AE$ , can be drawn from the point  $A$  which does not cut the circle.

Wherefore, *the straight line, etc.*

[Q. E. D.

**Corollaries.** (1) The straight line which is drawn at right angles to the diameter of a circle from the extremity of it touches the circle.

[III. Definition 2.

(2) A tangent touches the circle at one point only, because if it did meet the circle at two points it would fall within it.

[III. 2.

(3) There can be but one tangent at the same point.

### EXERCISES.

1. Draw parallel to a given straight line a straight line to touch a given circle.

2. Draw perpendicular to a given straight line a straight line to touch a given circle.

3. Two circles have the same centre. Show that all chords of the outer circle which touch the inner circle are equal, and are bisected at the point of contact.

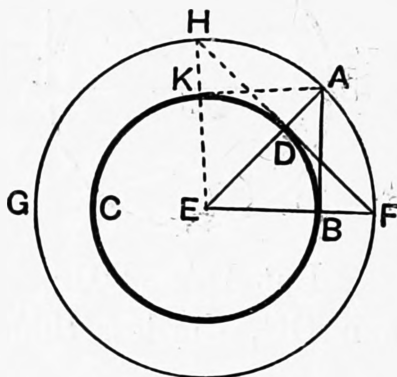


## PROPOSITION 17. PROBLEM.

*From a given point, either without or on the circumference, to draw a tangent to a given circle.*

CASE I. Let the given point A be without the given circle BCD :

*it is required to draw from A a tangent to the given circle.*



**Construction.** Find E, the centre of the circle, [III. 1.  
and join AE, cutting the given circle at D ; with centre E  
and distance EA describe the circle AFG ; from D draw  
HDF at right angles to EA to meet the circle AFG in F  
and H, [I. 11.  
and join EF and EH, cutting the given circle at B and K ;  
join AB and AK. AB and AK shall touch the circle BCD.

**Proof.** In the triangles AEB, FED,  
because  $\begin{cases} AE = FE, \text{ being radii of the outer circle,} \\ \text{and } EB = ED, \text{ being radii of the given circle,} \\ \text{and the angle AEB is common ;} \end{cases}$   
 $\therefore$  the triangles are equal in all respects ; [I. 4.  
 $\therefore$  the  $\angle ABE =$  the  $\angle FDE$ .

But the  $\angle FDE$  is a right  $\angle$  ; [Construction.  
 $\therefore$  the  $\angle ABE$  is a right  $\angle$ . [Axiom 1.

$\therefore$  BA, being drawn at right angles to a diameter from one of  
its extremities B, touches the circle. [III. 16, Corollary.

Similarly it can be shewn that AK touches the circle at K.

Also AB and AK are drawn from the given point A. [Q.E.F.

CASE II. If the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; then DF touches the circle. [III. 16, Cor.

**Corollaries.** (1) Two, and only two, tangents can be drawn to a circle from an external point.

(2) These two tangents are equal, subtend equal angles at the centre, and make equal angles with the line joining the centre to the given external point.

For each of the angles ABE, AKE is a right angle;

$\therefore$  the squares on AB, BE = the square on AE,

and the squares on AK, KE = the square on AE.

But the square on BE = the square on KE;

$\therefore$  the square on AB = the square on AK;  $\therefore$  AK = AB,

and the triangles AKE, ABE are equal in all respects;

$\therefore$  the  $\angle KAE$  = the  $\angle BAE$ , and

the  $\angle KEA$  = the  $\angle BEA$ .

### EXERCISES.

**\*\*1.** Prove that the following construction also gives the tangents from A. Join EA and bisect it in O; with centre O and radius OA or OE, describe a circle and let it meet the given circle in P and Q; join AP and AQ: these are the required tangents.

[Join OP; since  $OP = OE$ ,  $\therefore \angle OEP = \angle OPE$ ; since  $OP = OA$ ,  $\therefore \angle OAP = \angle OPA$ ;  $\therefore \angle^s OEP, OAP = \angle^s OPE, OPA = \angle APE$ . Hence, by I. 32,  $\angle APE$  is a right angle, and AP is a tangent at P; so AQ is a tangent at Q.]

**\*\*2.** The centre of any circle which touches two given straight lines lies on the bisector of the angle between them.

**\*\*3.** A quadrilateral is described so that its sides touch a circle. Shew that two of its opposite sides are together equal to the other two sides, and conversely. [Use Cor. 2.]

**4.** If a hexagon, or any polygon having an even number of sides, circumscribe a circle, the sums of its alternate sides are equal.

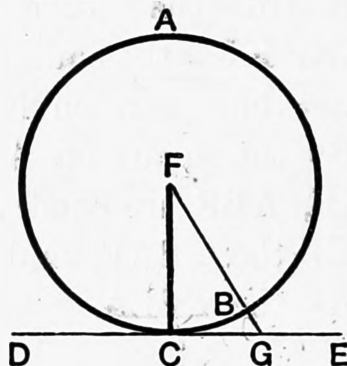
**\*\*5.** CN is drawn from the centre C of a circle perpendicular to the chord AB. Prove that the tangents at A and B meet at a point T on CN produced.

Prove also that the rectangle CN, CT equals the square on CA.

## PROPOSITION 18. THEOREM.

*If a straight line touch a circle, the radius drawn from the centre of the point of contact shall be perpendicular to the line touching the circle.*

Let the straight line DE touch the circle ABC at the point C;  
the radius FC, drawn from the centre F, shall be perpendicular to DE.



**Construction.** For if not, let FG be drawn from F perpendicular to DE, meeting the circumference at B.

**Proof.** In the  $\triangle FCG$ , because  $\angle FGC$  is a right  $\angle$ , [*Hypoth.* the  $\angle FCG$  is less than a right  $\angle$  ; [I. 17.

$\therefore$  FC is greater than FG. [I. 19.

But  $FC = FB$  ; [I. Definition 15.

$\therefore$  FB is greater than FG, the less than the greater, which is impossible.

$\therefore$  FG is not perpendicular to DE.

Similarly, it may be shewn that no other straight line from F is perpendicular to DE except FC ;

$\therefore$  FC is perpendicular to DE.

Wherefore, *if a straight line, etc.*

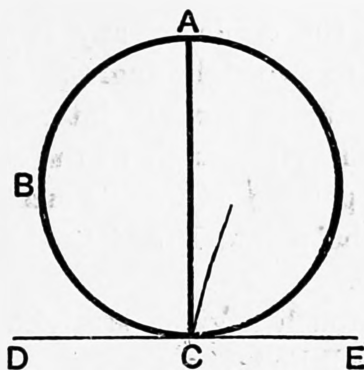
[Q.E.D.]

## PROPOSITION 19. THEOREM.

*If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.*

Let the straight line DE touch the circle ABC at C, and from C let CA be drawn at right angles to DE: the centre of the circle shall be in CA.

For, if not, if possible, let F be the centre, and join CF.



**Proof.** Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular to DE; [III. 18.]

$\therefore$  the  $\angle FCE$  is a right  $\angle$ .

But the  $\angle ACE$  is also a right  $\angle$ ; [Construction.]

$\therefore$  the  $\angle FCE =$  the  $\angle ACE$ , [Axiom 11.]

that is, the less to the greater, which is impossible.

$\therefore$  F is not the centre of the circle ABC.

Similarly, it may be shewn that no other point out of CA is the centre; therefore the centre is in CA.

Wherefore, if a straight line, etc.

[Q. E. D.]

## EXERCISES.

**\*\*1.** If two tangents to a circle be parallel, the points of contact are at the extremities of a diameter.

**\*\*2.** Shew that no parallelogram can be described about a circle except a rhombus.

[Let ABCD be a  $\square^m$  described about a circle, centre O, and let E, F, G, H be the points of contact of the sides AB, BC, CD, DA.

By Ex. 1, EOG, FOH are straight lines ;

$\therefore$  by III. 16 Cor., OA, OC bisect the  $\angle^s$  EOH, FOG, and are  $\therefore$  in a straight line ;

$\therefore \triangle^s$  CGO, AOE are equal in all respects, so that  $CG = AE = AH$  ;

$\therefore DC = DG + GC = DH + HA = DA$ , etc.]

**\*\*3.** If a quadrilateral be described about a circle, the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

**4.** Two parallel tangents to a circle intercept on a third tangent a distance which subtends a right angle at the centre.

**5.** The tangent at any point A of a circle meets two fixed tangents in P and Q. Prove that the points P and Q subtend a constant angle at the centre of the circle.

**6.** A series of circles touch a given straight line at a given point. Where will their centres all lie ?

**7.** Describe a circle of given radius to pass through a given point and touch a given straight line.

**8.** A straight line is drawn touching two circles. Shew that the chords are parallel which join the points of contact and the points where the straight line through the centres meets the circumferences.

**9.** If two tangents TP, TQ be drawn to a circle, prove that the angle between them is double the angle between the line PQ joining their points of contact, and the diameter through either of them.

**10.** In the diameter of a circle produced, determine a point so that the tangent drawn from it to the circumference shall be of given length.

**11.** Two circles touch externally in A and the straight line BC touches them in B and C. Prove that BAC is a right angle.

**12.** Through a given point within or without a given circle draw a chord of given length not greater than the diameter.

[Use Page 137, Ex. 3.]

**\*\*13.** Draw a straight line to touch each of two given circles.

[See App., Art. 39.]

**14.** Construct a triangle, given the vertical angle, one of the sides containing it, and the perpendicular altitude.

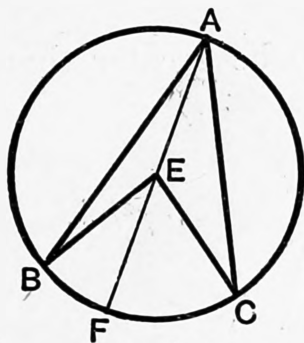
## PROPOSITION 20. THEOREM.

*The angle at the centre of a circle is double of the angle at the circumference on the same arc.*

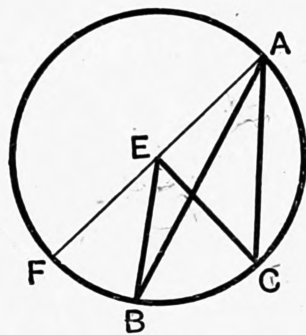
Let  $ABC$  be a circle, and  $BEC$  an angle at the centre, and  $BAC$  an angle at the circumference, which have the same arc  $BC$  for their base :

*the angle  $BEC$  shall be double of the angle  $BAC$ .*

**Construction.** Join  $AE$ , and produce it to  $F$ .



CASE I.



CASE II.

**Proof.** CASE I. Let the centre of the circle be within the angle  $BAC$ .

Then, because  $EA = EB$ , the  $\angle EAB = \text{the } \angle EBA$ ; [I. 5.]

$\therefore$  the angles  $EAB, EBA$  are double of the angle  $EAB$ .

But the angle  $BEF = \text{the angles } EAB, EBA$ ; [I. 32.]

$\therefore$  the  $\angle BEF$  is double of the  $\angle EAB$ .

Similarly, the  $\angle FEC$  is double of the  $\angle EAC$ .

$\therefore$  the whole  $\angle BEC$  is double of the whole  $\angle BAC$ .

CASE II. Let the centre  $E$  be without the  $\angle BAC$ .

As in Case I., the  $\angle FEC$  is double of the  $\angle FAC$ , and the  $\angle FEB$ , a part of the first, is double of the  $\angle FAB$ , a part of the other ;

$\therefore$  the remaining  $\angle BEC$  is double of the remaining  $\angle BAC$ .

CASE III. If the centre  $E$  lie on the straight line  $AB$  it is clear, if the corresponding figure be drawn, that, as in Case I., the  $\angle BEC$  is double of the  $\angle BAC$ .

Wherefore, *the angle at the centre, etc.*

[Q.E.D.]

[For an important note on III. 20, see Page 326.]

## PROPOSITION 21. THEOREM.

*The angles in the same segment of a circle are equal to one another.*

Let ABCD be a circle, and BAD, BED angles in the same segment BAED:

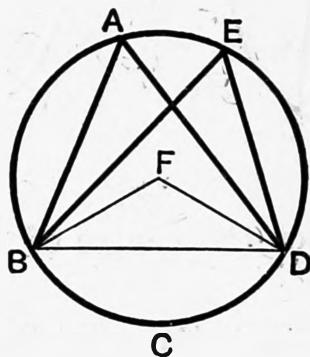
*the angles BAD, BED shall be equal to one another.*

**Construction.** Find F the centre of the circle ABCD.

[III. 1.]

CASE I. Let the segment BAED be greater than a semi-circle.

Join BF, DF.



**Proof.** Because the angles BFD and BAD are angles at the centre and circumference standing on the same arc BD.

$\therefore$  the  $\angle BFD$  is double the  $\angle BAD$ . [III. 20.]

Similarly the  $\angle BFD$  is double the  $\angle BED$ .

$\therefore$  the  $\angle BAD =$  the  $\angle BED$ . [Axiom 7.]

CASE II. Let the segment BAED be not greater than a semicircle.

**Construction.** Draw AF to the centre, and produce it to meet the circumference at C, and join CE.

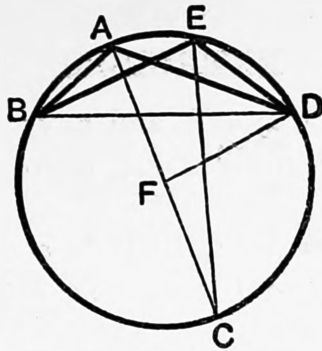
**Proof.** Since CFD and CED are angles at the centre and circumference standing on the same arc CD;

$\therefore$  the  $\angle CFD$  is double the  $\angle CED$ ; [III. 20.]

Similarly, the  $\angle CFD$  is double the  $\angle CAD$ ;

$\therefore$  the  $\angle CAD =$  the  $\angle CED$ .





But the segment BAEC is greater than a semicircle, and therefore the  $\angle BAC = \angle BEC$ , by Case I.

$\therefore$  the whole  $\angle BAD = \angle BED$ .

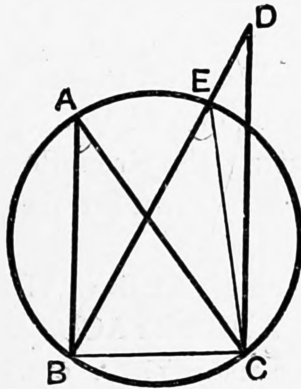
[Axiom 2.

Wherefore, *the angles in the same segment, etc.*

[Q.E.D.]

### CONVERSE OF PROPOSITION 21.

*The vertices of equal angles, standing on the same base and on the same side of it, lie on a segment of a circle passing through the extremities of the base.*



Let two of the angles be  $\angle ABC$  and  $\angle ADC$ .  
Describe a circle about ABC.

[This may be done by Prop. 5 of Book IV., which the student is now recommended to read.]

If this circle do not pass through D, let it cut BD in E, and join EC.  
Then  $\angle BEC = \angle BAC$ , by the preceding proposition,  
 $= \angle BDC$ , by hypothesis.

But this is impossible by I. 16,

$\therefore$  D must lie on the circle through B, A, C.

Similarly the vertex of any other such equal angle, whose arms pass through B and C, must lie on this same circle.



## EXERCISES.

1. Two circles cut at A and B, and any chord PAQ is drawn terminated by the two circumferences; prove that the angle PBQ is constant.

2. If two straight lines AEB, CED in a circle intersect at E, the angles subtended by AC and BD at the centre are together double of the angle AEC.

3. Two tangents AB, AC are drawn to a circle; D is any point on the circumference outside of the triangle ABC; shew that the sum of the angles ABD and ACD is constant.

4. P, Q are any points in the circumferences of two segments described on the same straight line AB, and on the same side of it; the angles PAQ, PBQ are bisected by the straight lines AR, BR meeting at R; shew that the angle ARB is constant.

$$[\angle PAR = \angle RAQ, \text{ i.e. } \angle PAB - \angle RAB = \angle RAB - \angle QAB;$$

$$\therefore 2\angle RAB = \angle PAB + \angle QAB.$$

$$\text{So } 2\angle RBA = \angle PBA + \angle QBA;$$

$$\therefore 2\angle^s RAB, RBA = \angle^s PAB, PBA, QAB, QBA,$$

$$\text{i.e. } 4 \text{ rt. } \angle^s - 2\angle ARB = 2 \text{ rt. } \angle^s - \angle APB + 2 \text{ rt. } \angle^s - \angle AQB;$$

$$\therefore 2\angle ARB = \angle APB + \angle AQB = \text{const.}]$$

5. Two segments of a circle are on the same base AB, and P is any point in the circumference of one of the segments; the straight lines APD, BPC are drawn meeting the circumference of the other segment at D and C; AC and BD are drawn intersecting at Q; shew that the angle AQB is constant.

$$[\angle AQB = \angle ACB - \angle CBQ, \text{ and } \angle APB = \angle ADB + \angle CBQ$$

$$\therefore \angle^s AQB, APB = \angle^s ACB, ADB = 2\angle ACB.$$

But the  $\angle^s APB, ACB$  are constant;  $\therefore$  etc.]

6. APB is a fixed chord passing through P a point of intersection of two circles AQP, PBR, and QPR is any other chord of the circles passing through P; shew that AQ and RB when produced meet at a constant angle.

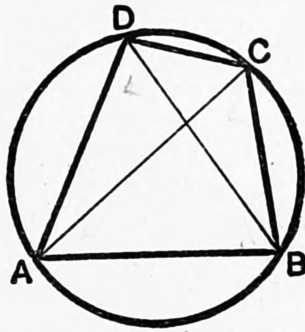
7. Two circles meet in A and B; through A two chords ACD, AC'D' are drawn cutting the circles in C, D and C', D'; prove that the  $\triangle^s BCD, BC'D'$  are equiangular.

## PROPOSITION 22. THEOREM.

*The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.*

Let ABCD be a quadrilateral figure inscribed in the circle ABCD :

*any two of its opposite angles shall be together equal to two right angles.*



**Construction.** Join AC, BD.

**Proof.** The  $\angle CAB =$  the  $\angle CDB$ , because they are in the same segment CDAB ; [III. 21.]

and the  $\angle ACB =$  the  $\angle ADB$ , because they are in the same segment ADCB ;

$\therefore$  the two angles CAB, ACB together = the whole  $\angle ADC$ .

[Axiom 2.]

To each of these equals add the  $\angle ABC$  ;

$\therefore$  the three angles CAB, ACB, ABC = the two angles ABC, ADC.

But the three angles CAB, ACB, ABC, being the angles of the  $\triangle ACB$ , are together equal to two right angles ; [I. 32.]

$\therefore$  also the angles ABC, ADC together = two right angles.

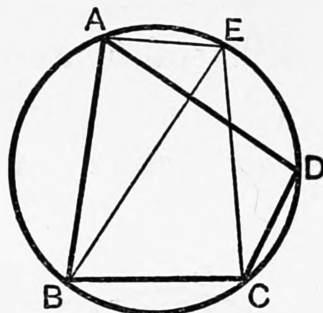
In the same manner it may be shewn that the angles BAD, BCD are together equal to two right angles.

Wherefore, *the opposite angles, etc.*

[Q. E. D.]

## CONVERSE OF III. 22.

This is true and very important ; namely,  
*If two opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral.*



For, let ABCD denote the quadrilateral. Describe a circle round the triangle ABC, by IV. 5. Take any point E on the arc cut off by AC and on the same side of AC that D is.

Then the angles at B and E together = two right angles, by III. 22, and the angles at B and D together = two right angles, by hypothesis ;

$\therefore$  the angle at E = the angle at D ;

$\therefore$  by the converse of III. 21, D is on the same segment as E.

*Note.* A quadrilateral, such as ABCD in the above figure, which can be inscribed in a circle is called a **Cyclic Quadrilateral**, and the four points A, B, C, D are said to be **Concyclic**.

## EXERCISES.

**\*\*1.** If one side of a quadrilateral inscribed in a circle be produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

**2.** The straight lines which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle meet on the circumference of the circle. [Use Ex. 1.]

**3.** A triangle is inscribed in a circle ; shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

**4.** A quadrilateral is inscribed in a circle ; shew that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

**5.** If a polygon of an even number of sides be inscribed in a circle, the sum of the alternate angles together with two right angles is equal to as many right angles as the figure has sides.

[Let  $ABCDEF\dots$  be the polygon and  $O$  the centre of the circle.  
 The  $\angle ABC = 2 \text{ rt. } \angle^s - \frac{1}{2} \angle AOC$ ;  $\angle CDE = 2 \text{ rt. } \angle^s - \frac{1}{2} \angle COE$ , etc.;  
 $\therefore$  on addition, the sum of the alternate angles  
 $= 2 \text{ rt. } \angle^s \times \frac{1}{2} \text{ no. of the sides} - \frac{1}{2} \text{ all the } \angle^s \text{ at } O = \text{etc.}]$

**6.** Shew that the four straight lines bisecting the interior (or the exterior) angles of any quadrilateral form a quadrilateral which can be inscribed in a circle.

[Let the bisectors of the interior angles  $A$  and  $B$ ,  $B$  and  $C$ ,  $C$  and  $D$ ,  $D$  and  $A$  of a quadrilateral meet in  $E$ ,  $F$ ,  $G$ ,  $H$ . Then

$$\angle FEH = 2 \text{ rt. } \angle^s - \frac{1}{2} \angle A - \frac{1}{2} \angle B,$$

$$\text{and } \angle FGH = 2 \text{ rt. } \angle^s - \frac{1}{2} \angle C - \frac{1}{2} \angle D;$$

$$\therefore \angle^s FEH, FGH = 4 \text{ rt. } \angle^s - \text{half the sum of the interior angles of } ABCD \\ = 4 \text{ rt. } \angle^s - 2 \text{ rt. } \angle^s = 2 \text{ rt. } \angle^s; \therefore \text{etc.}$$

Similarly for the bisectors of the exterior angles.]

**\*\*7.** Shew that no parallelogram except a rectangle can be inscribed in a circle.

**8.**  $D$  is any point on the arc  $BC$  of a circle whose centre is  $A$ ;  $CD$  is produced to  $E$ ; prove that the angle  $BDE$  is half the angle  $BAC$ .

**9.**  $AOB$  is a triangle;  $C$  and  $D$  are points in  $BO$  and  $AO$  respectively, such that the angle  $ODC$  is equal to the angle  $OBA$ ; shew that a circle may be described round the quadrilateral  $ABCD$ .

**10.**  $ABCD$  is a quadrilateral inscribed in a circle, and the sides  $AB$ ,  $DC$  when produced meet at  $O$ ; shew that the triangle  $AOC$  is equiangular to the triangle  $BOD$ , and the triangle  $AOD$  to the triangle  $BOC$ .

**\*\*11.** If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

[Let  $ABCDEF$  be the hexagon having  $FA$ ,  $AB$  parallel to  $CD$ ,  $DE$ . Then  $\angle FEB = 2 \text{ rt. } \angle^s - \angle FAB$  [III. 22]  $= 2 \text{ rt. } \angle^s - \angle DEC$  [I. 29]  $= \angle EBC$ .

**12.**  $ABCD$  is a quadrilateral inscribed in a circle;  $AB$  and  $CD$  meet in  $P$ ;  $AD$  and  $BC$  meet in  $Q$ ; prove that the bisectors of the angles  $P$  and  $Q$  are at right angles.

[If  $O$  be the intersection of the bisectors, and  $OQ$  meet  $CD$  in  $L$ , then

$$\angle OPC = \frac{1}{2} \angle D - \frac{1}{2} \angle PAD \text{ and } \angle OQD = \frac{1}{2} \angle D - \frac{1}{2} \angle DCQ;$$

$$\therefore \angle POQ = \angle PLQ - \angle OPC = \angle D - \angle OQD - \angle OPC$$

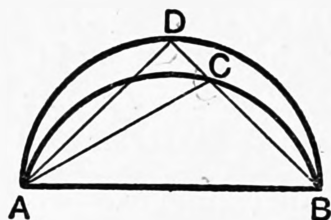
$$= \angle D - \frac{1}{2} (2 \angle D - \angle PAD - \angle PCQ)$$

$$= \frac{1}{2} \angle PAD + \frac{1}{2} \angle PCQ = a \text{ rt. } \angle.]$$

## PROPOSITION 23. THEOREM.

*On the same chord, and on the same side of it, there cannot be two similar segments of circles which do not coincide with one another.*

If it be possible, on the same chord  $AB$ , and on the same side of it, let there be two similar segments of circles  $ACB$ ,  $ADB$  not coinciding with one another.



**Construction.** Because the circle  $ACB$  cuts the circle  $ADB$  at the two points  $A$ ,  $B$ , they cannot cut one another at any other point; [III. 10.]

therefore one of the segments must fall within the other;

let  $ACB$  fall within  $ADB$ ;

draw the straight line  $BCD$ , and join  $AC$ ,  $AD$ .

**Proof.** Because  $ACB$ ,  $ADB$  are, by hypothesis, similar segments of circles, they contain equal angles; [III. Definition 11.]

$\therefore$  the  $\angle ACB =$  the  $\angle ADB$ ,

that is, the exterior  $\angle$  of the  $\triangle ACD$  is equal to the interior and opposite  $\angle$ , which is impossible. [I. 16.]

Wherefore, *on the same straight line, etc.*

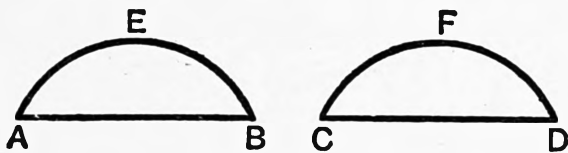
[Q. E. D.]

## PROPOSITION 24. THEOREM.

*Similar segments of circles on equal chords are equal to one another.*

Let AEB, CFD be similar segments of circles on the equal chords AB, CD :

*the segment AEB shall be equal to the segment CFD.*



**Proof.** If the segment AEB be applied to the segment CFD, so that the point A may be on the point C, and AB on CD,

then B will coincide with D, because  $AB = CD$  ; [Hyp.

$\therefore$  AB coinciding with CD, the segment AEB must coincide with the segment CFD, [III. 23.

and is therefore equal to it.

Wherefore, *similar segments, etc.*

[Q. E. D.

## PROPOSITION 25. PROBLEM.

*A segment of a circle being given, to describe the circle of which it is a segment.*

Let  $ABC$  be the given segment of a circle :  
*it is required to describe the circle of which it is a segment.*

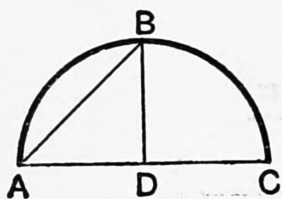


FIG. 1.

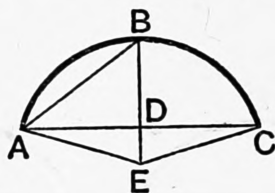


FIG. 2.

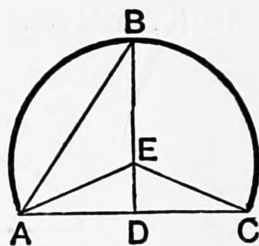


FIG. 3.

**Construction.** Bisect  $AC$  at  $D$  ; [I. 10.]  
 from  $D$  draw  $DB$  at right angles to  $AC$  to meet the circumference in  $B$ , [I. 11.]  
 and join  $AB$ .

CASE I. Let the angles  $ABD$ ,  $BAD$  be equal [Fig 1].

Then  $DB = DA$  ; [I. 6.]

but  $DA = DC$  ; [Construction.]

therefore  $DB = DC$ . [Axiom 1.]

$\therefore DA, DB, DC$  are all equal,

and therefore  $D$  is the centre of the circle. [III. 9.]

With  $D$ , and radius equal to any of the three  $DA, DB, DC$ , describe a circle ; this will pass through the other points, and the circle of which  $ABC$  is a segment is described.

Also because the centre  $D$  is in  $AC$ , the segment  $ABC$  is a semicircle.

CASE II. Let the angles  $ABD, BAD$  be not equal [Figs. 2, 3].

**Construction.** At the point  $A$  make the angle  $BAE$  equal to the angle  $ABD$  ; [I. 23.]

produce  $BD$ , if necessary, to meet  $AE$  at  $E$ , and join  $EC$ .

**Proof.** Because the  $\angle BAE =$  the  $\angle ABE$ , [Construction.  
 $EA = EB$ . [I. 6.

Then, in the triangles ADE, CDE,

because  $\begin{cases} AD = CD, \\ \text{and } DE \text{ is common,} \\ \text{and the right } \angle ADE = \text{the right } \angle CDE; \end{cases}$  [Construction.  
 $\therefore$  the base  $EA =$  the base  $EC$ . [I. 4.

But  $EA$  was shewn to be equal to  $EB$ ;  
 therefore  $EB = EC$ . [Axiom 1.

$\therefore$  the three straight lines  $EA$ ,  $EB$ ,  $EC$  are all equal,  
 and therefore a circle, whose centre is  $E$  and whose radius is  
 any of the three  $EA$ ,  $EB$ ,  $EC$ , will pass through the other  
 points, and the circle of which  $ABC$  is a segment is described.  
 [III. 9.

And it is evident that, if the angle  $ABD$  be greater than the  
 angle  $BAD$ , the centre  $E$  falls without the segment  $ABC$ ,  
 which is therefore less than a semicircle;  
 but if the angle  $ABD$  be less than the angle  $BAD$ , the centre  
 $E$  falls within the segment  $ABC$ , which is greater than a  
 semicircle.

Wherefore *a segment of a circle being given, the circle has been  
 described of which it is a segment.* [Q.E.F.

#### ALTERNATIVE METHOD FOR PROP. 25.

The following is an alternative method: *Let A, B, C be three points  
 on the arc; join AB and BC; bisect AB and BC in D and E respec-  
 tively; draw DF perpendicular to AB and EF perpendicular to BC,  
 and let them meet in F. Then F shall be the centre of the required  
 circle.*

For by III. 1 Cor. the centre is in the straight line  $DF$ , which bisects  
 $AB$  at right angles.

Similarly, it is in  $EF$ .

$\therefore$  it is at  $F$ , and a circle whose centre is at  $F$  and radius equal to  
 either  $FA$ ,  $FB$ , or  $FC$  will be the circle required; for as in III. 1,  
 $FA = FB$  and  $FB = FC$ .

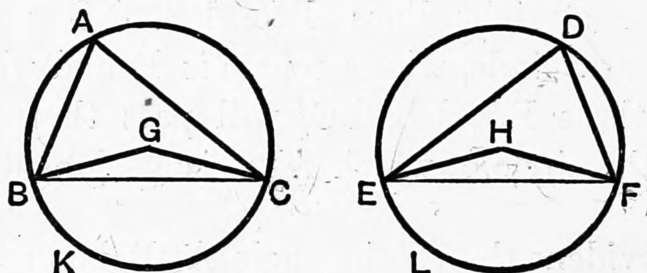


## PROPOSITION 26. THEOREM.

*In equal circles the arcs which are subtended by equal angles, whether they be at the centres or circumferences, are equal.*

Let  $ABC$ ,  $DEF$  be equal circles; and let  $BGC$ ,  $EHF$  be equal angles in them at their centres, and  $BAC$ ,  $EDF$  equal angles at their circumferences:

*the arc  $BKC$  shall be equal to the arc  $ELF$ .*



**Construction.** Join  $BC$ ,  $EF$ .

**Proof.** Because the circles  $ABC$ ,  $DEF$  are equal, [Hyp.  
their radii are equal; [III. Definition 1.

Then, in the triangles  $BGC$ ,  $EHF$ ,

because  $\begin{cases} BG = EH, \\ \text{and } GC = HF, \\ \text{and the } \angle BGC = \text{the } \angle EHF; \end{cases}$  [Hypothesis.  
 $\therefore$  the base  $BC =$  the base  $EF$ . [I. 4.

And because the  $\angle BAC =$  the  $\angle EDF$ , [Hypothesis.  
the segment  $BAC$  is similar to the segment  $EDF$ , [III. Def. 11.  
and they are on equal chords  $BC$ ,  $EF$ .

$\therefore$  the segment  $BAC =$  the segment  $EDF$ . [III. 24.

But the whole circle  $ABC =$  the whole circle  $DEF$ ;

[Hypothesis.  
 $\therefore$  the remaining segment  $BKC =$  the remaining segment  $ELF$ ; [Axiom 3.

$\therefore$  the arc  $BKC =$  the arc  $ELF$ .

Wherefore, in equal circles, etc.

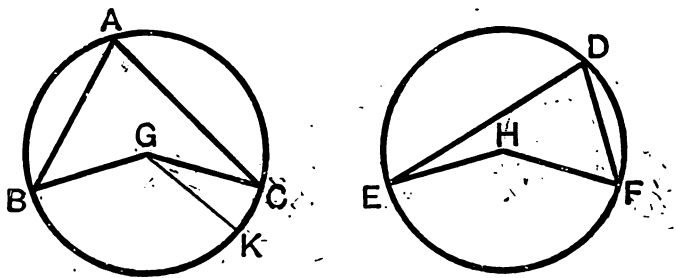
[Q.E.D.]

## PROPOSITION 27. THEOREM.

*In equal circles, the angles which stand on equal arcs are equal to one another, whether they be at the centres or circumferences.*

Let  $ABC$ ,  $DEF$  be equal circles, and let the angles  $BGC$ ,  $EHF$  at their centres, and the angles  $BAC$ ,  $EDF$  at their circumferences, stand on equal arcs  $BC$ ,  $EF$ :

*the angle  $BGC$  shall be equal to the angle  $EHF$ , and the angle  $BAC$  equal to the angle  $EDF$ .*



If the  $\angle BGC$  be equal to the  $\angle EHF$ , it is manifest that the  $\angle BAC$  is also equal to the  $\angle EDF$ . [III. 20, *Axiom 7*.

But, if not, one of them must be the greater.

Let  $BGC$  be the greater, and at  $G$  make the  $\angle BGK$  equal to the  $\angle EHF$ . [I. 23.

**Proof.** Because the  $\angle BGK =$  the  $\angle EHF$ ;

$\therefore$  the arc  $BK =$  the arc  $EF$ . [III. 26.

But the arc  $EF =$  the arc  $BC$ ; [*Hypothesis*,

$\therefore$  the arc  $BK =$  the arc  $BC$ , [*Axiom 1*.

the less to the greater, which is impossible.

$\therefore$  the  $\angle BGC$  is not unequal to the  $\angle EHF$ ,  
that is, it is equal to it.

And the  $\angle$  at  $A$  is half of the  $\angle BGC$ , and the  $\angle$  at  $D$  is half of the  $\angle EHF$ ; [III. 20.

$\therefore$  the  $\angle$  at  $A =$  the  $\angle$  at  $D$ . [*Axiom 7*.

Wherefore, *in equal circles, etc.* [Q.E.D.

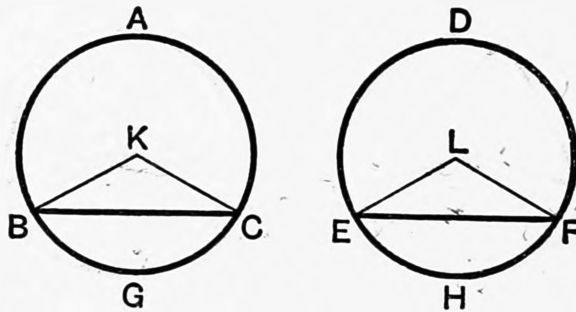
*Note.* Prop. 27 is the converse of Prop. 26.

## PROPOSITION 28. THEOREM.

*In equal circles, the arcs which are cut off by equal chords are equal, the greater equal to the greater, and the less equal to the less.*

Let  $ABC$ ,  $DEF$  be equal circles, and  $BC$ ,  $EF$  equal chords in them, which cut off the two greater arcs  $BAC$ ,  $EDF$ , and the two less arcs  $BGC$ ,  $EHF$ :

*the greater arc  $BAC$  shall be equal to the greater arc  $EDF$ , and the less arc  $BGC$  equal to the less arc  $EHF$ .*



**Construction.** Find  $K$ ,  $L$ , the centres of the circles,

[III. 1.]

and join  $BK$ ,  $KC$ ,  $EL$ ,  $LF$ .

**Proof.** Because the circles are equal,  
their radii are equal;

[Hypothesis.]

[III. Definition 1.]

Then, in the triangles  $BKC$ ,  $ELF$ ,

because  $\left\{ \begin{array}{l} BK = EL, \\ \text{and } KC = LF, \\ \text{and the base } BC = \text{the base } EF, \end{array} \right.$

[Hypothesis.]

$\therefore$  the  $\angle BKC = \text{the } \angle ELF$ ; [I. 8.]

$\therefore$  the arc  $BGC = \text{the arc } EHF$ . [III. 26.]

But the circumference  $ABGC = \text{the circumference } DEHF$ ;

[Hypothesis.]

$\therefore$  the remaining arc  $BAC = \text{the remaining arc } EDF$ . [Ax. 3.]

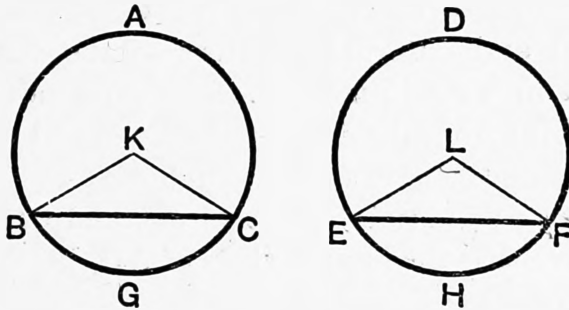
Wherefore, *in equal circles, etc.*

[Q. E. D.]

## PROPOSITION 29. THEOREM.

*In equal circles, the chords which cut off equal arcs are equal.*

Let  $ABC$ ,  $DEF$  be equal circles, and let  $BGC$ ,  $EHF$  be equal arcs in them, and join  $BC$ ,  $EF$ :  
*the chord  $BC$  shall be equal to the chord  $EF$ .*



**Construction.** Find  $K$ ,  $L$ , the centres of the circles, [III. 1.  
 and join  $BK$ ,  $KC$ ,  $EL$ ,  $LF$ .

**Proof.** Because the arc  $BGC =$  the arc  $EHF$ , [*Hypothesis.*  
                   the angle  $BKC =$  the angle  $ELF$ . [III. 27.  
 And because the circles  $ABC$ ,  $DEF$  are equal, [*Hypothesis.*  
                   their radii are equal; [III. Definition 1.  
 $\therefore$  the two sides  $BK$ ,  $KC =$  the two sides  $EL$ ,  $LF$ ,  
                   and they contain equal angles; [Proved.  
 $\therefore$  the base  $BC =$  the base  $EF$ . [I. 4.  
 Wherefore, *in equal circles, etc.* [Q.E.D.

*Note 1.* Prop. 29 is the converse of Prop. 28.

*Note 2.* The Propositions 26–29 tell us that in equal circles  
 If two angles are equal, so also are the chords and arcs subtended;  
 If two chords are equal, so also are the angles subtended and  
   the arcs cut off;  
 If two arcs are equal, so also are the angles and chords.

*Note 3.* It is clear that Props. 26–29 could easily be proved by  
*Superposition.*

**EXERCISES.**

**1.** The straight lines joining the extremities of the chords of two equal arcs of a circle, towards the same parts, are parallel to each other.

**2.** The straight lines in a circle which join the extremities of two parallel chords are equal to each other.

**\*3.** The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, all intersect at the same point.

**4.** The internal and external bisectors of the vertical angle of a triangle inscribed in a circle meet the circumference again in points which are equidistant from the ends of the base, and which lie on the straight line bisecting the base at right angles.

**5.** If A, B be two fixed points on a circle, and CD a chord of constant length ; prove that the intersections of AD, BC and of AC, BD lie on two fixed circles.

[Since CD is constant in length,

$\therefore$  the  $\angle^s$  CAD, CBD are constant [Props. 28, 27].

Hence if AD, BC meet in O, then

$$\angle AOB = \angle ACB + \angle DAC = \text{const.} ; \therefore \text{etc.}]$$

**6.** AB is a common chord of two circles ; through C any point of one circumference straight lines CAD, CBE are drawn terminated by the other circumference ; shew that the arc DE is invariable.

**7.** If two equal circles cut each other, and if through one of the points of intersection a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

**8.** The straight line joining the feet of the perpendiculars from any point of a circle upon two diameters given in position is constant.

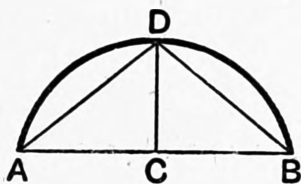
Conversely, if a straight line of given length move so that its ends slide along two given straight lines, the two perpendiculars at its ends to the given lines meet on a fixed circle.

## PROPOSITION 30. PROBLEM.

*To bisect a given arc, that is, to divide it into two equal parts.*

Let  $ADB$  be the given arc :

*it is required to bisect it.*



**Construction.** Join  $AB$ , and bisect it at  $C$  ; [I. 10.  
from the point  $C$  draw  $CD$  at right angles to  $AB$  meeting  
the arc at  $D$ . [I. 11.

The arc  $ADB$  shall be bisected at the point  $D$ .

Join  $AD$ ,  $DB$ .

**Proof.** In the triangles  $ACD$ ,  $BCD$ ,

because  $\begin{cases} AC = CB, & \text{[Construction.]} \\ \text{and } CD \text{ is common to both,} \\ \text{and the right } \angle ACD = \text{the right } \angle BCD ; & \text{[Const.]} \end{cases}$   
 $\therefore$  the base  $AD =$  the base  $BD$ . [I. 4.

But equal straight lines cut off equal arcs, the greater equal  
to the greater, and the less equal to the less, [III. 23.

and each of the arcs  $AD$ ,  $DB$  is less than a semicircle,

because  $DC$ , if produced, is a diameter ; [III. 1, Cor.

$\therefore$  the arc  $AD =$  the arc  $DB$ .

Wherefore, *the given arc is bisected at  $D$ .*

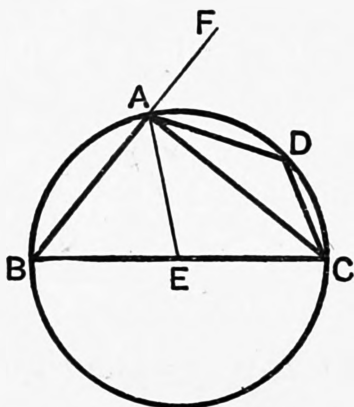
[Q. E. F.]

## PROPOSITION 31. THEOREM.

*An angle in a semicircle is a right angle,  
but an angle in a segment greater than a semicircle is less than a  
right angle,  
and an angle in a segment less than a semicircle is greater than  
a right angle.*

Let ABCD be a circle, of which BC is a diameter and E the centre ; and draw CA, dividing the circle into the segments ABC, ADC :

- (1) *the angle in the semicircle BAC shall be a right angle ;*  
 (2) *the angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle ;*  
 (3) *the angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.*



**Construction.** Join AE, and produce BA to F.  
Take any point D on the arc ADC, and join AD, DC.

**Proof.** (1) Because  $EA = EB$ , [I. Definition 15.  
the  $\angle EAB = \text{the } \angle EBA$  ; [I. 5.

and, because  $EA = EC$ ,  
the  $\angle EAC = \text{the } \angle ECA$  ;

$\therefore$  the whole angle  $BAC = \text{the two angles } ABC, ACB$ . [Ax. 2.

But  $FAC$ , the exterior angle of the triangle  $ABC$ ,  
= the two angles  $ABC, ACB$  ; [I. 32.

$\therefore$  the  $\angle BAC = \text{the } \angle FAC$ , [Axiom 1.

and therefore each of them is a right  $\angle$  ; [I. Definition 10.

$\therefore$  the  $\angle$  in a semicircle  $BAC$  is a right  $\angle$ .

(2) Because the two angles  $ABC$ ,  $BAC$  of the triangle  $ABC$  are together less than two right angles, [I. 17.] and that  $BAC$  has been shewn to be a right  $\angle$ ,  
 $\therefore$  the  $\angle ABC$  is less than a right  $\angle$  ;  
 $\therefore$  the  $\angle$  in a segment  $ABC$ , greater than a semicircle, is less than a right  $\angle$ .

(3) Because  $ABCD$  is a quadrilateral figure in a circle, any two of its opposite angles together = two right angles ; [III. 22.]  
 $\therefore$  the angles  $ABC$ ,  $ADC$  together = two right angles.  
 But the  $\angle ABC$  has been shewn to be less than a right  $\angle$ .  
 $\therefore$  the  $\angle ADC$  is greater than a right  $\angle$  ;  
 $\therefore$  the  $\angle$  in a segment  $ADC$ , less than a semicircle, is greater than a right  $\angle$ .

Wherefore, *the angle, etc.*

[Q.E.D.]

**Corollary.** *If one angle of a triangle be equal to the other two, it is a right angle.*

For the angle adjacent to it is equal to the same two angles ; [I. 32.] and when the adjacent angles are equal, they are right angles.  
 [I. Definition 10.]

### EXERCISES.

**\*\*1.** Right-angled triangles are described on the same hypotenuse ; shew that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

**2.** The circles described on the sides of any triangle as diameters, will intersect on the base.

**3.**  $AOB$  and  $COD$  are two perpendicular diameters of a circle. If  $P$  be any point on the circumference, prove that  $CP$  and  $DP$  are the internal and external bisectors of the angle  $APB$ .

**4.** Chords  $AB$ ,  $CD$  of a circle cut one another at right angles ; the sum of the opposite arcs  $AC$  and  $BD$  is a semicircle.



5. AB is the diameter of a circle whose centre is C, and DCE is a sector having the arc DE constant; join AE, BD intersecting at P; shew that the angle APB is constant.

6. On the side AB of any triangle ABC as diameter a circle is described; EF is a diameter parallel to BC; shew that the straight lines EB and FB bisect the interior and exterior angles at B.

7. If AD, CE be drawn perpendicular to the sides BC, AB of a triangle ABC, and DE be joined, shew that the angles ADE and ACE are equal. [A, E, D, C lie on a circle.]

8. If two circles ABC, ABD intersect at A and B, and AC, AD be two diameters, shew that the straight line CD will pass through B.

9. If O be the centre of a circle and OA a radius, and a circle be described on OA as diameter, the circumference of this circle will bisect any chord of the exterior circle drawn from A.

10. If from the angles at the base of any triangle perpendiculars are drawn to the opposite sides, produced if necessary, the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the centre of the base.

11. If two chords of a circle meet at right angles within or without a circle, the squares on their segments are together equal to the squares on the diameter.

[Let the chords be AB, CD meeting within the circle at O and let AA' be the diameter through A. Then

$$\angle CAB = \text{rt. } \angle - \angle ACD = \text{rt. } \angle - \angle AA'D = \angle DAA';$$

$$\therefore DA' = BC;$$

$$\therefore AO^2 + OD^2 + BO^2 + OC^2 = AD^2 + BC^2 = AD^2 + DA'^2 = AA'^2.$$

Similarly if the chords cut without the circle.]

12. AB is a diameter of a circle and C a given point in AB; find a point in the circumference at which both AC and CB will subtend half a right angle.

[Join C to the middle point D of the arc AB; the required point is the point where CD meets the circle again.]

13. Circles are described on the sides of a quadrilateral as diameters; prove that the common chord of two of these circles which are adjacent is parallel to the common chord of the other two.

14. ABCD is a quadrilateral figure inscribed in a circle; if the bisectors of two of its opposite angles meet the circle in E and F, then EF is a diameter.

15. Divide a straight line into two parts such that the rectangle contained by the two parts may be equal to a given square. When is this impossible? [Use II. 5.]

## PROPOSITION 32. THEOREM.

*If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent shall be equal to the angles which are in the alternate segments of the circle.*

Let the straight line  $EF$  touch the circle  $ABCD$  at the point  $B$ , and from  $B$  let the chord  $BD$  be drawn, cutting the circle : then

(1) *the angle  $DBF$  shall be equal to the angle in the segment  $BAD$ , and*

(2) *the angle  $DBE$  shall be equal to the angle in the segment  $BCD$ .*

**Construction.** From  $B$  draw  $BA$  at right angles to  $EF$ , [I. 11.] and take any point  $C$  in the arc  $BD$ .

Join  $AD$ ,  $DC$ ,  $CB$ .

**Proof.** (1) Because  $BA$  is drawn at right angles to the tangent  $EF$  at its point of contact  $B$ ,

[Constr.

∴ the centre of the circle is in  $BA$ ; [III. 19.]

∴ the  $\angle ADB$ , being in a semicircle, is a right  $\angle$ ; [III. 31.]

∴ the other two angles  $BAD$ ,  $ABD$  together = a right  $\angle$ . [I. 32.]

But  $ABF$  is also a right angle; [Constr.]

∴ the  $\angle ABF$  = the angles  $BAD$ ,  $ABD$ .

From each of these equals take the common  $\angle ABD$ ;

∴ the remaining  $\angle DBF$  = the remaining  $\angle BAD$ , [Ax. 3.]

which is in the alternate segment of the circle.

(2) Because  $ABCD$  is a quadrilateral in a circle, the angles  $BAD$ ,  $BCD$  together = two right angles. [III. 22.]

But the angles  $DBF$ ,  $DBE$  together = two right angles; [I. 13.]

∴ the angles  $DBF$ ,  $DBE$  together = the angles  $BAD$ ,  $BCD$ .

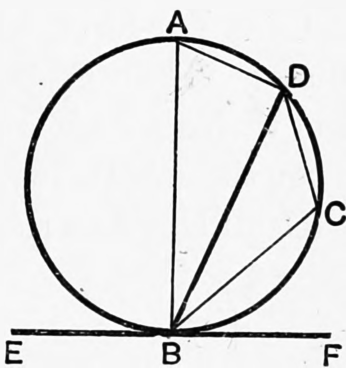
And the  $\angle DBF$  has been shewn equal to the  $\angle BAD$ ;

∴ the remaining  $\angle DBE$  = the remaining  $\angle BCD$ , [Ax. 3.]

which is in the alternate segment of the circle.

Wherefore, *if a straight line, etc.*

[Q.E.D.]

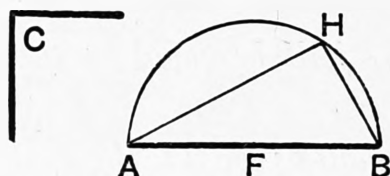


## PROPOSITION 33. PROBLEM.

*On a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.*

Let  $AB$  be the given straight line, and  $C$  the given rectilineal angle :

*it is required to describe on  $AB$  a segment of a circle containing an angle equal to  $C$ .*



CASE I. First, let the  $\angle C$  be a right  $\angle$ .

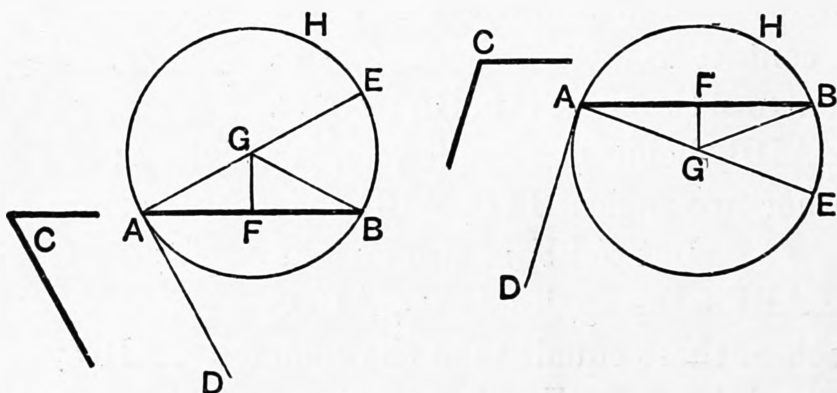
Bisect  $AB$  at  $F$ ,

[I. 10.]

and with centre  $F$  and radius  $FB$  describe the semicircle  $AHB$ .

Then the  $\angle AHB$  in a semicircle is equal to the right  $\angle C$ . [III. 31.]

CASE II. Let the  $\angle C$  be not a right  $\angle$ .



**Construction.** At  $A$  make the  $\angle BAD$  equal to the  $\angle C$ ; [I. 23.]

from  $A$  draw  $AE$  at right angles to  $AD$ ;

[I. 11.]

bisect  $AB$  at  $F$ ;

[I. 10.]

from  $F$  draw  $FG$  at right angles to  $AB$  to meet  $AE$  in  $G$ . [I. 11.]

Join  $GB$ .

**Proof.** In the triangles  $AFG$ ,  $BFG$ ,

because  $\begin{cases} AF = BF, \\ \text{and } FG \text{ is common,} \\ \text{and the right } \angle AFG = \text{the right } \angle BFG; \end{cases}$  [I. Ax. 11.]

$\therefore$  the base  $AG =$  the base  $BG$ , [I. 4.  
and therefore the circle described from the centre  $G$ , with  
radius  $GA$ , will pass through the point  $B$ .

Let this circle be described ; and let it be  $AHB$ .

The segment  $AHB$  shall contain an angle equal to the given  
rectilineal angle  $C$ .

Because  $AD$  is drawn at right angles to the diameter  $AE$ ,  
 $\therefore AD$  touches the circle. [III. 16. *Corollary*.

Also, because  $AB$  is drawn from the point of contact  $A$ ,  
the  $\angle DAB =$  the  $\angle$  in the alternate segment  $AHB$ . [III. 32.

But the  $\angle DAB =$  the  $\angle C$ ; [Construction.

$\therefore$  the  $\angle$  in the segment  $AHB =$  the  $\angle C$ . [Axiom 1.

Wherefore, *on the given straight line  $AB$ , the segment  $AHB$  of  
a circle has been described, containing an angle equal to the given  
angle  $C$ .* [Q.E.F.

## EXERCISES ON PROPOSITION 32.

1. Prove the converse of III. 32. [See Notes, page 327.]

\*\*2. If a tangent to a circle is parallel to a chord, the point of contact  
of the tangent will be the middle point of the arc cut off by the chord.

3. Two circles touch each other externally. Prove that any straight  
line drawn through the point of contact cuts off similar segments from  
the two circles.

4. If two circles intersect one another, prove that each common  
tangent subtends at either common point angles that are equal or  
supplementary.

5.  $B$  is a point in the circumference of a circle whose centre is  $C$ ;  $PA$ ,  
tangent at any point  $P$ , meets  $CB$  produced at  $A$ , and  $PD$  is drawn per-  
pendicular to  $CB$ ; shew that the straight line  $PB$  bisects the angle  $APD$ .

6. Two circles intersect at  $A$  and  $B$ , and through  $P$  any point in the  
circumference of one of them the chords  $PA$  and  $PB$  are drawn to cut  
the other circle at  $C$  and  $D$ ; shew that  $CD$  is parallel to the tangent  
at  $P$ .

**7.** If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars dropped on them from the middle point of the intercepted arc are equal.

**8.** AB is a chord and AD a tangent to a circle at A. DPQ is a straight line parallel to AB, meeting the circle in P and Q. Prove that the triangles PAD, QAB are equiangular.

**9.** On a straight line AB as base, and on the same side of it are described two segments of circles; P is any point in the circumference of one of the segments, and the straight line BP cuts the circumference of the other segment at Q; shew that the angle PAQ is equal to the angle between the tangents at A.

**10.** C is the centre of a circle; CA, CB are two radii at right angles; from B any chord BP is drawn cutting CA at N: a circle being described round ANP, shew that it will be touched by BA.

**11.** AB and CD are parallel straight lines, and the straight lines which join their extremities intersect at E: shew that the circles described round the triangles ABE, CDE touch one another.

**12.** If the centres of two circles which touch each other externally be fixed, the common tangent of those circles will touch another circle of which the straight line joining the fixed centres is the diameter.

### EXERCISES ON PROPOSITION 33.

**1.** Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls.

[On the given base AB describe a segment of a circle containing the given vertical angle; by the converse of III. 22 the vertex lies somewhere on this segment. Through the given point in the base draw a straight line perpendicular to the base; this will meet the segment in the required vertex. There are two solutions, since two segments can be drawn, one on each side of AB.]

**2.** Construct a triangle, having given the base, the vertical angle, and the altitude.

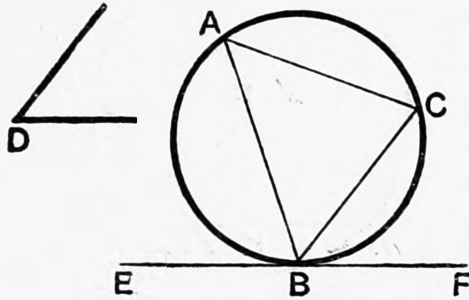
**3.** Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the middle point of the base.

**4.** Having given the base and the vertical angle of a triangle, shew that the triangle will be greatest when it is isosceles.

## PROPOSITION 34. PROBLEM.

*From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.*

Let  $ABC$  be the given circle, and  $D$  the given angle :  
*it is required to cut off from the circle  $ABC$  a segment containing an angle equal to  $D$ .*



**Construction.** Draw the straight line  $EF$  touching the circle  $ABC$  at  $B$  ; [III. 17.  
 and at  $B$  make the  $\angle FBC$  equal to the  $\angle D$ . [I. 23.  
 The segment  $BAC$  shall contain an  $\angle$  equal to  $D$ .

**Proof.** Because  $EF$  touches the circle  $ABC$ , and  $BC$  is drawn from the point of contact  $B$  ; [Construction.  
 $\therefore$  the  $\angle FBC =$  the  $\angle$  in the alternate segment  $BAC$ . [III. 32.  
 But the  $\angle FBC =$  the  $\angle D$  ; [Construction.  
 $\therefore$  the  $\angle$  in the segment  $BAC =$  the  $\angle D$ . [Axiom 1.

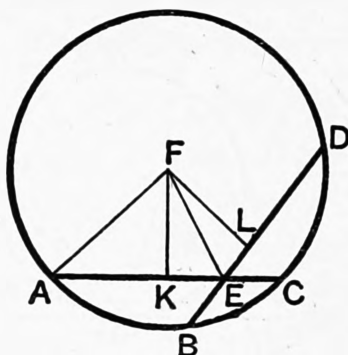
Wherefore, from the given circle  $ABC$ , the segment  $BAC$  has been cut off, containing an angle equal to the given angle  $D$ . [Q.E.F.

## PROPOSITION 35. THEOREM.

*If two chords of a circle cut one another within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*

Let the two chords AC, BD cut one another at the point E within the circle ;

*the rectangle AE, EC shall be equal to the rectangle BE, ED.*



**Construction.** CASE I. Find the centre F, and, if it be not in AC or BD, join FE.

Draw FK perpendicular to AC and FL perpendicular to BD. Join FA.

**Proof.** Since AC is bisected at K and divided unequally at E,  
 $\therefore$  the rect. AE, EC together with the square on KE  
 $=$  the square on AK. [II. 5.]

To each add the square on FK ,  
 $\therefore$  the rect. AE, EC, together with the squares on FK, KE,  
 $=$  the squares on AK, KF.

But, since the angles at K are right angles, [Construction,

the squares on FK, KE  $=$  the square on FE, [I. 47.]

and the squares on AK, KF  $=$  the square on FA ; [I. 47.]

$\therefore$  the rect. AE, EC together with the square on FE  
 $=$  the square on FA.

Similarly, the rect. BE, ED together with the square on FE  
 $=$  the square on FD, that is, the square on FA ;

$\therefore$  the rect. AE, EC with the square on FE

$=$  the rect. BE, ED with the square on FE.

Take away the common square on FE ;

$\therefore$  the rect. AE, EC  $=$  the rect. BE, ED.



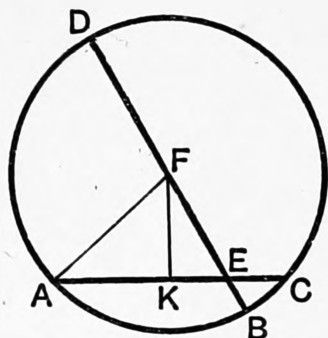
CASE II. Next, let the centre  $F$  be on one of the chords  $BD$ , and let  $BD$  be not perpendicular to  $AC$ .

Then since  $BD$  is bisected at  $F$  and divided unequally at  $E$ ,

$\therefore$  the rect.  $BE$ ,  $ED$  with the square on  $EF$  = the square on  $FB$ ,

[II. 5.

that is, the square on  $FA$ .

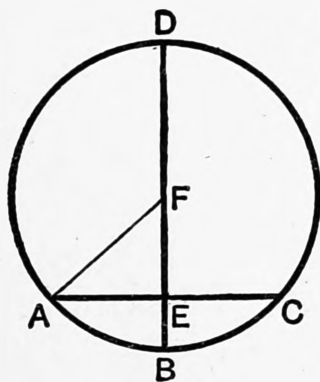


Also, as in Case I.,

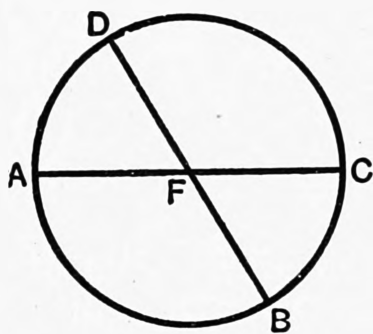
the rect.  $AE$ ,  $EC$  with the square on  $FE$  = the square on  $FA$ ;

$\therefore$  as before, the rect.  $AE$ ,  $EC$  = the rect.  $BE$ ,  $ED$ .

CASE III. Next, let  $F$  be on  $BD$ , and let  $BD$  be perpendicular to  $AC$ . Then  $BD$  will bisect  $AC$ ;



CASE III.



CASE IV.

$\therefore$  the rect.  $AE$ ,  $EC$  = the square on  $AE$ ,

and  $\therefore$  the rect.  $AE$ ,  $EC$ , together with the square on  $EF$ ,  
= the squares on  $AE$ ,  $EF$ ,

that is, = the square on  $FA$ , that is, as in the second case,

= the rect.  $BE$ ,  $ED$  with the square on  $EF$ ,

and thus the rect.  $AE$ ,  $EC$  = the rect.  $BE$ ,  $ED$

CASE IV. Lastly, let both chords pass through the centre  $F$ .

In this case the lines  $AF$ ,  $FC$ ,  $BF$ ,  $FD$  are all equal;

$\therefore$  the rect.  $AF$ ,  $FC$  = the rect.  $BF$ ,  $FD$ ;

for each is equal to the square on the radius of the circle.

Wherefore, *if two chords, etc.*

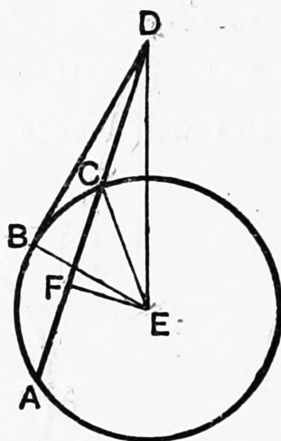


## PROPOSITION 36. THEOREM.

*If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.*

From any point D without the circle ABC let there be drawn two straight lines DCA, DB, of which DCA cuts the circle and DB touches it:

*the rectangle AD, DC shall be equal to the square on DB.*



**Construction.** CASE I. Let DCA not pass through the centre of the circle ABC; find the centre E; [III. 1.  
from E draw EF perpendicular to AC; [I. 12.  
and join EB, EC, ED.

**Proof.** Because EF drawn from the centre cuts the chord AC at right angles at F,  $\therefore AF = FC$ . [III. 3.

And because AC is bisected at F, and produced to D, the rectangle AD, DC, together with the square on FC,  
= the square on FD. [II. 6.

To each of these equals add the square on FE;

$\therefore$  the rect. AD, DC, with the squares on CF, FE,

= the squares on DF, FE. [Axiom 2.

But, because CFE is a right angle, the squares on CF, FE

= the square on CE, [I. 47.

that is, = the square on BE;

and the squares on  $DF$ ,  $FE$  = the square on  $DE$ ;  
 $\therefore$  the rect.  $AD$ ,  $DC$ , together with the square on  $BE$ ,  
 = the square on  $DE$ .

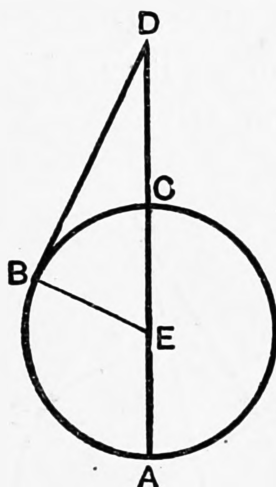
that is, = the squares on  $DB$ ,  $BE$ ,  
 because  $EBD$  is a right angle.

[I. 47.]

Take away the common square on  $BE$ ;

then the remaining rect.  $AD$ ,  $DC$  = the square on  $DB$ . [Axiom 3.]

CASE II. If  $DCA$  passes through  $E$ , join  $EB$ .



Since  $AC$  is bisected at  $E$  and produced to  $D$ ,  
 the rect.  $AD$ ,  $DC$ , with the square on  $EC$ ,  
 = the square on  $ED$ ,

[II. 6.]

that is, = the sqs. on  $EB$ ,  $BD$ , since  $EBD$  is a right  $\angle$ . [I. 47.]

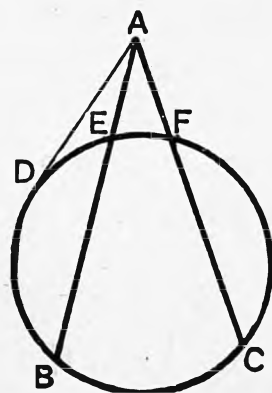
But the square on  $EC$  = the square on  $EB$ ;

$\therefore$  the rect.  $AD$ ,  $DC$  = the square on  $BD$ .

Wherefore, *if from any point, etc.*

[Q.E.D.]

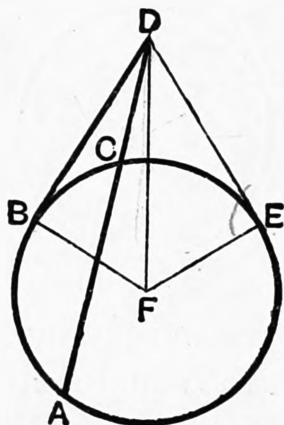
**Corollary.** If from any point without a circle there be drawn secants  $AEB$ ,  $AFC$ , the rectangles contained by the whole secants and the parts of them without the circles are equal, that is, the rectangle  $BA$ ,  $AE$  = the rectangle  $CA$ ,  $AF$ ; for each of them is equal to the square on the tangent  $AD$ .



## PROPOSITION 37. THEOREM.

*If from any point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall be a tangent.*

Let any point  $D$  be taken without the circle  $ABC$ , and from it let two straight lines  $DCA$ ,  $DB$  be drawn, of which  $DCA$  cuts the circle, and  $DB$  meets it; and let the rectangle  $AD$ ,  $DC$  be equal to the square on  $DB$ :  
 $DB$  shall touch the circle.



**Construction.** Draw the straight line  $DE$ , touching the circle  $ABC$ ; [III. 17.]

find  $F$  the centre, [III. 1.]  
 and join  $FB$ ,  $FD$ ,  $FE$ .

**Proof.** Because  $DE$  touches the circle  $ABC$ , and  $DCA$  cuts it,  
 the rect.  $AD$ ,  $DC$  = the square on  $DE$ . [III. 36.]  
 But the rect.  $AD$ ,  $DC$  = the square on  $DB$ ; [Hypothesis.]  
 $\therefore$  the square on  $DE$  = the square on  $DB$ . [Axiom 1.]

Hence, in the triangles  $DBF$ ,  $DEF$ ,

because  $\begin{cases} DE = DB, \\ \text{and } EF = BF, \\ \text{and the base } DF \text{ is common,} \end{cases}$

I. Definition 15.

$\therefore$  the  $\angle DEF$  = the  $\angle DBF$ .

[I. 8.]

But DEF is a right  $\angle$ , since DE is a tangent ; [*Constr.* and III. 18.

$\therefore$  also DBF is a right  $\angle$ .

And BF, if produced, is a diameter ;

$\therefore$  DB touches the circle ABC.

[III. 16, *Corollary*.

Wherefore, *if from a point, etc.*

[Q. E. D.]

### EXERCISES.

**\*\*1.** Prove the following converse of III. 35 and III. 36, Cor. : *If two straight lines AB, CD intersect in O and the rectangle AO, OB = the rectangle CO, OD, the four points A, B, C, D lie on a circle.*

[For if the circle passing through A, B, C do not pass through D, let it meet CD in E ; then, by III. 35 or III. 36 Cor. (according as O is within or without CD) the rectangle CO, OE = the rectangle AO, OB = the rectangle CO, OD (Hyp.),  $\therefore$  OE = OD. Hence E coincides with D, and thus the circle through A, B, C passes through D.]

**\*\*2.** If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

**\*\*3.** Two circles intersect. Shew that their common chord produced bisects their common tangent.

**4.** If AD, CE are drawn perpendicular to the sides BC, AB of a triangle ABC, shew that the rectangle contained by BC and BD is equal to the rectangle contained by BA and BE.

**5.** If through any point in the common chord of two circles which intersect one another, there be drawn any two other chords, one in each circle, their four extremities all lie on the circumference of a circle.

**6.** From a given point as centre, describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.

[The radius of the circle is such that its square is equal to the difference of the square on the line joining the two given points and the given square.]

**7.** A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal. Shew that the straight lines joining the two points of intersection of each pair will pass through this point.

## BOOK IV.

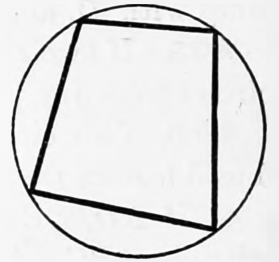
### DEFINITIONS.

1. A rectilineal figure is said to be **inscribed** in another rectilineal figure when all the angles of the inscribed figure are on the sides of the figure in which it is inscribed, each on each.



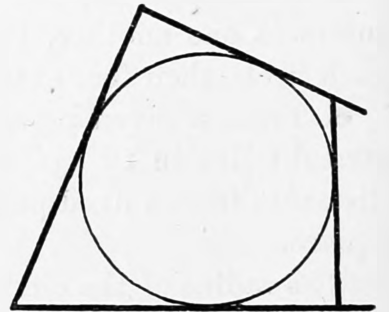
2. In like manner a figure is said to be **described about**, or **circumscribed about**, another figure when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

3. A rectilineal figure is said to be **inscribed** in a circle when all the angles of the inscribed figure are on the circumference of the circle.



4. A rectilineal figure is said to be **described about** a circle when each side of the circumscribed figure touches the circumference of the circle.

5. In like manner a circle is said to be **inscribed** in a rectilineal figure when the circumference of the circle touches each side of the figure.



Such a circle is often called the **in-circle** and its centre the **in-centre**.

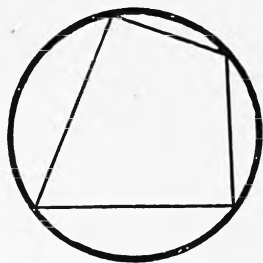
[A circle is said to be **escribed** to a triangle when it touches one side of the triangle and the other two sides produced. For a figure, see page 181.]

6. A circle is said to be described about a rectilineal figure when the circumference of the circle passes through all the angular points of the figure about which it is described.

Such a circle is often called the **circum-circle** and its centre the **circum centre**.

7. A straight line is said to be placed in a circle when the extremities of it are on the circumference of the circle.

8. A polygon is a rectilineal figure contained by more than four straight lines. [I. Def. 22.



A polygon of five sides is called a **pentagon** (Gk. πέντε).

„ six „ „ **hexagon** (Gk. ἑξ).

„ seven „ „ **heptagon** (Gk. ἑπτά).

„ eight „ „ an **octagon** (Gk. ὀκτώ).

„ ten „ „ a **decagon** (Gk. δέκα).

„ twelve „ „ **dodecagon** (Gk. δώδεκα).

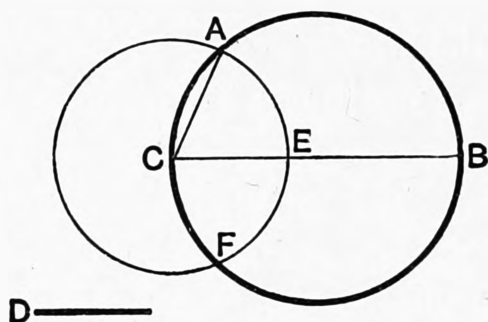
„ fifteen „ „ **quindecagon** (Lat. *quinque*; Gk. δέκα).

9. A **regular polygon** is one which has all its sides equal and also all its angles equal, *i.e.* it is equilateral and equiangular.

## PROPOSITION 1. THEOREM.

*In a given circle, to draw a chord equal to a given straight line, which is not greater than the diameter of the circle.*

Let  $ABC$  be the given circle, and  $D$  the given straight line, not greater than the diameter of the circle :  
*it is required to draw a chord of the circle  $ABC$  equal to  $D$ .*



**Construction.** Draw  $BC$ , a diameter of the circle  $ABC$ .  
 Then, if  $BC = D$ , the thing required is done ; for, in the circle  $ABC$ , a straight line is placed equal to  $D$ .

But, if it is not,  $BC$  is greater than  $D$ .

[*Hypothesis.*

Make  $CE$  equal to  $D$ ,

[*I. 3.*

and with centre  $C$  and radius  $CE$  describe the circle  $AEF$   
 and join  $CA$ .

**Proof.** Because  $C$  is the centre of the circle  $AEF$ ,

$$CA = CE ;$$

[*I. Definition 15.*

$$\text{but } CE = D ;$$

[*Construction.*

$$\therefore CA = D.$$

[*Axiom 1.*

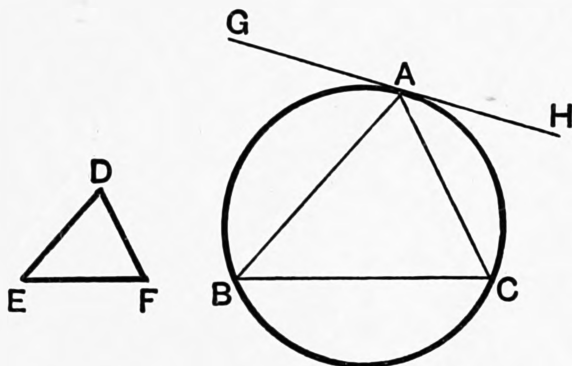
Wherefore, *in the circle  $ABC$ , a straight line  $CA$  is placed equal to the given straight line  $D$ , which is not greater than the diameter of the circle.*

[*Q. E. F.*

## PROPOSITION 2. PROBLEM.

*In a given circle, to inscribe a triangle equiangular to a given triangle.*

Let  $ABC$  be the given circle, and  $DEF$  the given triangle: *it is required to inscribe in the circle  $ABC$  a triangle equiangular to  $DEF$ .*



**Construction.** At any point  $A$  on the circumference draw the tangent  $GAH$ ; [III. 17.]  
 at  $A$  make the  $\angle HAC$  equal to the  $\angle DEF$ , [I. 23.]  
 and also make the  $\angle GAB$  equal to the  $\angle DFE$ ;  
 join  $BC$ .  $ABC$  shall be the triangle required.

**Proof.** Because  $GAH$  touches the circle  $ABC$ , and  $AC$  is drawn from the point of contact  $A$ ; [Construction.]  
 $\therefore$  the  $\angle HAC =$  the  $\angle ABC$  in the alternate segment. [III. 32.]  
 But the  $\angle HAC =$  the  $\angle DEF$ ; [Construction.]  
 $\therefore$  the  $\angle ABC =$  the  $\angle DEF$ . [Axiom 1.]

Similarly, the  $\angle ACB =$  the  $\angle DFE$ ;  
 $\therefore$  the third  $\angle BAC =$  the third  $\angle EDF$ . [I. 32, Axioms 11 and 3.]

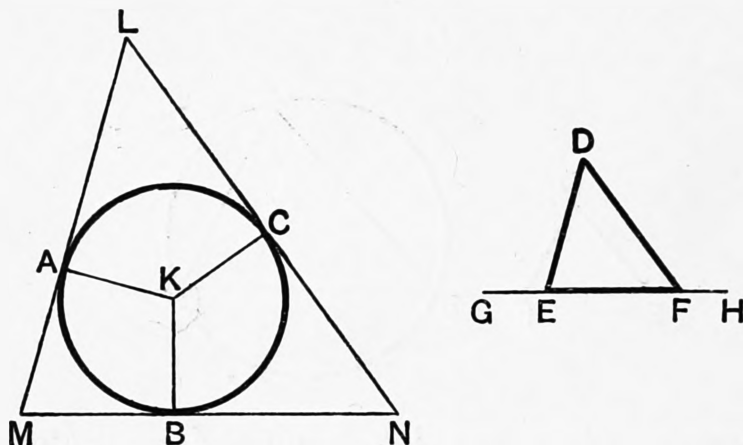
Wherefore the triangle  $ABC$  is equiangular to the triangle  $DEF$ , and it is inscribed in the circle  $ABC$ . [Q.E.F.]



## PROPOSITION 3. PROBLEM.

*About a given circle, to describe a triangle equiangular to a given triangle.*

Let  $ABC$  be the given circle, and  $DEF$  the given triangle :  
*it is required to describe a triangle about the circle  $ABC$  equiangular to the  $\triangle DEF$ .*



**Construction.** Produce  $EF$  both ways to  $G$  and  $H$  ;  
 find  $K$ , the centre of the circle  $ABC$  ; [III. 1.  
 from  $K$  draw any radius  $KB$  ;  
 at  $K$  make the  $\angle BKA$  equal to the  $\angle DEG$ ,  
                     and the  $\angle BKC$  equal to the  $\angle DFH$  ; [I. 23.  
 and through  $A, B, C$  draw the straight lines  $LAM, MBN, NCL$   
 at right angles to  $KA, KB,$  and  $KC$  respectively. [I. 11.  
 $LMN$  shall be the triangle required.

**Proof.** Because  $LM, MN, NL$  are drawn perpendicular to  
 the radii  $KA, KB, KC$  through their extremities, [*Construction.*  
                     therefore  $LM, MN, NL$  all touch the circle,  $\pi 16$ .  
                     and  $LMN$  is a  $\triangle$  described about the circle.

Also because the four angles of the figure  $AMBK$  are together  
 equal to four right angles,

for it can be divided into two triangles,

and two of them  $KAM, KBM$  are right angles,

$\therefore$  the other two  $AKB, AMB$  are together equal to two right  
 angles.

[*Axiom 3.*

But the angles DEG, DEF together = two right angles; [I.13.

$\therefore$  the angles AKB, AMB = the angles DEG, DEF;

of which the  $\angle$ AKB = the  $\angle$ DEG; [Construction.

$\therefore$  the remaining  $\angle$ AMB = the remaining  $\angle$ DEF. [Ax. 3.

Similarly, the angles LNM and DFE may be shewn to be equal;

$\therefore$  the third  $\angle$ MLN = the third  $\angle$ EDF. [I. 32, Axioms 11 and 3.

Wherefore *the triangle LMN is equiangular to the triangle DEF, and it is described about the circle ABC.* [Q.E.F.

### EXERCISES.

1. Place a chord in a given circle equal to a given straight line, so that it shall be parallel to another given straight line.

2. Place a chord of given length in a given circle, so that it may pass through a given point within or without the circle. When is this impossible?

3. Inscribe in a circle a triangle MNP whose sides are parallel to three given straight lines.

4. Two triangles are circumscribed to a given circle, each of them being equiangular to a given triangle; prove that the triangles are equal in all respects.

5. Any rectilinear figure ABCDE is inscribed in a circle; the arcs AB, BC, CD, DE, EA are bisected, and tangents drawn at the points of bisection; shew that the resulting figure is equiangular to ABCDE.

6. Prove that the area of an equilateral triangle inscribed in a circle is one-quarter that of the equilateral triangle circumscribed to the circle.

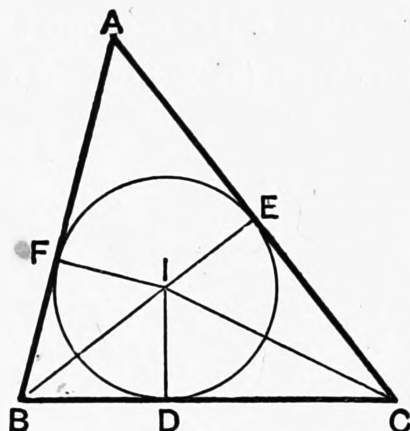
[Let ABC be an equilateral triangle inscribed in a given circle; at A, B, C draw tangents. Let the tangents at B, C meet in A'; those at C, A in B'; and those at A, B in C'. Then A'B'C' is an equilateral  $\triangle$  described about the circle, and it is easy to prove that the  $\triangle$ s A'CB, B'AC, C'BA and ABC are all equal.]

## PROPOSITION 4. PROBLEM.

*To inscribe a circle in a given triangle.*

Let  $ABC$  be the given triangle :

*it is required to inscribe a circle in the triangle  $ABC$ .*



**Construction.** Bisect the angles  $ABC$ ,  $ACB$  by the straight lines  $BI$ ,  $CI$ , meeting one another at the point  $I$ ; [I. 9.  
and from  $I$  draw  $ID$ ,  $IE$ ,  $IF$  perpendiculars to  $BC$ ,  $CA$ ,  $AB$ . [I. 12.]

**Proof.** In the triangles  $DBI$ ,  $FBI$ ,  
because  $\begin{cases} \text{the } \angle DBI = \text{the } \angle FBI, & [\text{Construction.}] \\ \text{and the right } \angle BDI = \text{the right } \angle BFI, \\ \text{and the side } BI \text{ is common;} \end{cases}$   
 $\therefore ID = IF$ . [I. 26.]

For the same reason  $ID = IE$ ;  
 $\therefore IE = IF$ ; [Axiom 1.]

$\therefore$  the three straight lines  $ID$ ,  $IE$ ,  $IF$  are equal and the circle described with centre  $I$ , and radius equal to any one of them, will pass through the extremities of the other two;  
and it will touch the straight lines  $BC$ ,  $CA$ ,  $AB$ , because the angles at the points  $D$ ,  $E$ ,  $F$  are right angles; [III. 16, Cor. 1.  
 $\therefore$  the circle  $DEF$  is inscribed in the triangle  $ABC$ .

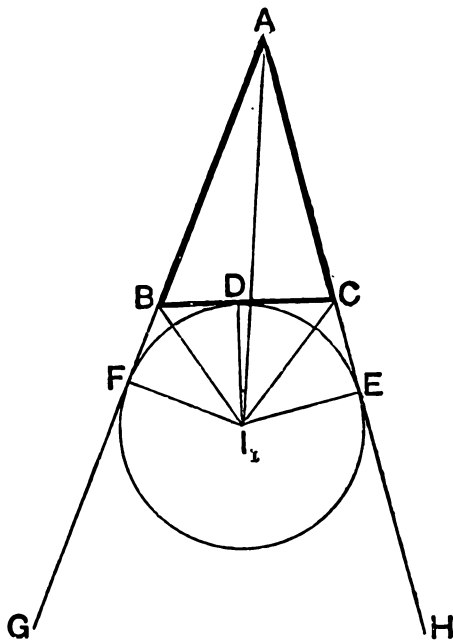
Wherefore a circle has been inscribed in the given triangle. [Q.E.F.]

## ADDITIONAL PROPOSITION.

*To draw an escribed circle of a given triangle.*

Let  $ABC$  be the given triangle :

*it is required to draw the escribed circle which is opposite to the angle  $A$ , that is, which touches  $BC$  externally.*



**Construction.** Produce  $AB$  and  $AC$  to  $G$  and  $H$  ;  
 bisect the exterior angles  $GBC$ ,  $HCB$  by the straight lines  $BI_1$ ,  $CI_1$ ,  
 which meet at  $I_1$  ;  
 draw  $I_1D$ ,  $I_1E$ , and  $I_1F$  perpendicular to  $BC$ ,  $CA$ , and  $AB$  respectively.

**Proof.** In the triangles  $I_1BD$ ,  $I_1BF$ ,  
 because  $\begin{cases} \text{the } \angle I_1BD = \text{the } \angle I_1BF, & [\text{Construction.}] \\ \text{and the right } \angle I_1DB = \text{the right } \angle I_1FB, \\ \text{and the side } I_1B \text{ is common ;} \end{cases}$   
 $\therefore$  the base  $I_1D = \text{the base } I_1F$ . [I. 26.]

Similarly, it may be shewn that  $I_1D = I_1E$  ;  
 therefore  $I_1D$ ,  $I_1E$ , and  $I_1F$  are equal, and a circle described with  
 centre  $I_1$ , and radius equal to either of these three, will pass through  
 the extremities of the other two.

Also, since the angles at  $D$ ,  $E$ ,  $F$  are right angles, this circle will touch  
 $BC$  and  $AB$ ,  $AC$  produced, and will therefore be the circle required.

**Corollary.** Since this circle touches  $AE$ ,  $AF$ , the angles  $I_1AE$  and  
 $I_1AF$  will be equal, [III. 17, Corollary.]  
 and  $I_1A$  will therefore bisect the angle  $BAC$ .

## EXERCISES.

**\*\*1.** In the figure of IV. 4, prove that AI bisects the angle BAC, and hence that the bisectors of the angles of a triangle meet in a point.

**\*\*2.** In the figure of IV. 4, prove that  $BD + CA = CE + AB = AF + BC$  = the semi-perimeter of the triangle ABC.

[ $BD = BF$ ,  $CD = CE$ ,  $AE = AF$  (III. 17, *Cor.*);

$\therefore$  sum of twice BD, CE, AE = sum of BD, BF, CE, CD, AE, AF  
= sum of BC, CA, AB;

$\therefore$  twice BD and twice AC = perimeter, etc.]

**3.** The circle inscribed in a triangle ABC touches the sides in the points D, E, and F. Prove that the angles of the triangle DEF are equal to the complements of half the angles of the triangle ABC, and hence that the triangle DEF is always acute-angled.

**4.** Without producing two straight lines to meet, find that straight line which would bisect the angle between them.

[Suppose BK, CL to be the two lines; draw any straight line BC to meet them; bisect  $\angle$ s at B, C by straight lines BI, CI meeting in I; draw IF, IE perpendicular to BK, CL and bisect FIE; this bisecting line will be the required line.]

**5.** With the vertices A, B, C of a triangle as centres draw three circles, each of which touches the other two.

**6.** A circle is inscribed in a triangle ABC, and a triangle is cut off at each angle by a tangent to the circle. Shew that the sides of the three triangles so cut off are together equal to the sides of ABC.

**7.** If the circle inscribed in a triangle ABC touch the sides AB, AC at the points D, E, prove that the middle point of the arc DE is the centre of the circle inscribed in the triangle ADE.

**8.** Find the centre of a circle cutting off three equal chords from the sides of a triangle.

The escribed circle opposite the angle A of a triangle touches the sides in the points  $D_1$ ,  $E_1$ ,  $F_1$  and the inscribed circle touches them in D, E, and F; prove that

**\*\*9.**  $AE_1 = AF_1$  = the semi-perimeter of the  $\triangle ABC$ .

**1c.**  $BD = CD_1$ .

**11.**  $DD_1$  = the difference between the sides AB and AC.

**12.** Two sides of a triangle whose perimeter is constant are given in position; prove that the third side always touches a certain circle.

I is the in-centre and  $I_1$ ,  $I_2$ ,  $I_3$  the centres of the escribed circles of the triangles ABC; prove that

**\*\*13.**  $AI I_1$ ,  $BI I_2$ , and  $CI I_3$  are straight lines. [See App. Art. 48.]

**\*\*14.**  $I_2 A I_3$ ,  $I_3 B I_1$ , and  $I_1 C I_2$  are straight lines.

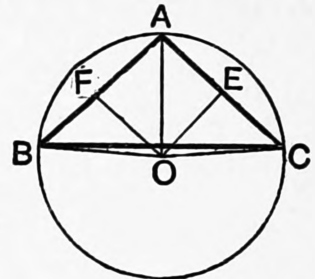
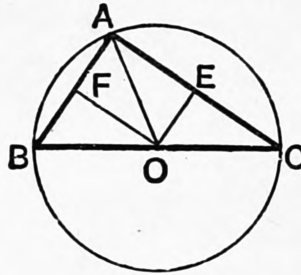
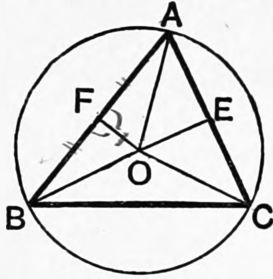
- \*\*15.**  $AII_1$  is perpendicular to  $I_2I_3$ , etc.
- 16.**  $I$ ,  $B$ ,  $C$ ,  $I_1$  lie on a circle.
- 17.**  $I_2$ ,  $I_3$ ,  $B$ ,  $C$  lie on a circle.
- 18.** the triangles  $BCI_1$ ,  $CAI_2$ , and  $ABI_3$  are equiangular.
- 19.** If the escribed circles opposite to the angles  $A$ ,  $B$ ,  $C$  of a triangle touch the sides  $BC$ ,  $CA$ ,  $AB$  in  $D_1$ ,  $D_2$ ,  $D_3$ , prove that  $AD_2 = BD_1$ ,  $BD_3 = CD_2$ , and  $CD_1 = AD_3$ .
- 20.** The triangle formed by joining the centres of the escribed circles of the triangle and that formed by joining the points of contact of the inscribed circle are equiangular.
- 21.** How many circles can, in general, be drawn to touch three straight lines? What are the exceptional cases?

## PROPOSITION 5. PROBLEM

*To describe a circle about a given triangle.*

Let  $ABC$  be the given triangle :

*it is required to describe a circle about  $ABC$ .*



**Construction.** Bisect  $AB$ ,  $AC$  at the points  $F$ ,  $E$ ; [I. 10.  
from these points draw  $FO$ ,  $EO$  at right angles to  $AB$ ,  $AC$ ;  
[I. 11.

$FO$ ,  $EO$ , produced, will meet one another ;  
for if they do not meet they are parallel ;  
therefore  $AB$ ,  $AC$ , which are at right angles to them, are  
parallel, which is absurd :

let them meet at  $O$ , and join  $OA$  ;  
also, if  $O$  be not in  $BC$ , join  $BO$ ,  $CO$ .

**Proof.** In the triangles  $AFO$ ,  $BFO$ ,  
because  $\begin{cases} AF = BF, & \text{[Construction.]} \\ \text{and } FO \text{ is common,} \\ \text{and the right } \angle AFO = \text{the right } \angle BFO ; \end{cases}$   
 $\therefore$  the base  $OA =$  the base  $OB$ . [I. 4.]

Similarly, it may be shewn that  $OC = OA$  ;

$\therefore OB = OC$ , [Axiom 1.]

and  $OA$ ,  $OB$ ,  $OC$  are all equal.

Therefore the circle described with centre  $O$ , and radius equal to any one of them, will pass through the extremities of the other two, and will be described about the triangle  $ABC$ .

Wherefore a circle has been described about the given triangle.

[Q.E.F.]

**Corollary.** It is clear that

(i.) when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle ;

(ii.) when the centre is in one of the sides of a triangle, the  $\angle$  opposite to this side, being in a semicircle, is a right  $\angle$  ; and

(iii.) when the centre falls without the triangle, the  $\angle$  opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right  $\angle$ . [III. 31.

Therefore, conversely, *if the given triangle be acute-angled, the centre of the circle falls within it ; if it be a right-angled triangle, the centre is in the side opposite to the right angle ; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.*

## EXERCISES.

**\*\*1.** In IV. 5, shew that the perpendicular from O on BC will bisect BC, and hence that the straight lines which bisect at right angles the sides of a triangle all meet in a point.

**\*\*2.** If the inscribed and circumscribed circles of a triangle be concentric, shew that the triangle must be equilateral.

The perpendicular from O on BC meets the circum-circle in K and L (K being on the opposite side of BC from A). Prove that

**3.**  $\angle BOK = \angle COK = A$ .

**4.** AK and AL bisect the interior and exterior angles at A.

**\*\*5.** K is the centre of the circum-circle of BIC, where I is the in-centre.

**6.** The angle between the radius of the circum-circle passing through the vertex A of a triangle ABC, and the perpendicular from A upon BC is equal to the difference of the base angles of the triangle.

**7.** O is the centre of the circle circumscribing a triangle ABC ; D, E, F the feet of the perpendiculars from A, B, C on the opposite



sides. Shew that  $OA$ ,  $OB$ ,  $OC$  are respectively perpendicular to  $EF$ ,  $FD$ ,  $DE$ .

**8.** In an equilateral triangle, prove that the radius of the circum-circle is twice that of the in-circle, and the radius of the escribed circle is three times that of the in-circle.

[Let  $O$  be the in- and circum-centre of the equilateral  $\triangle ABC$  and  $O_1$  the e-centre opposite  $A$ ; let  $AD$  be  $\perp^r$  to  $BC$  and  $P$  the middle point of  $OO_1$ . Then  $OCO_1$  is a rt.  $\angle$  and  $\therefore P$  is the circum-centre of  $\triangle OCO_1$ ;  $\therefore PO_1 = PC = PO$ ;  $\therefore \angle PCO = \angle POC = 2 \angle ACO = 2 \angle OCD = 2$  complement of  $\angle DCO_1 = 2 \angle DO_1C = \angle DPC$ , and  $\therefore POC$  is equilateral;  $\therefore OC = OP = 2 OD$ , and  $O_1D = O_1P + PD = 2 OD + OD = 3 OD$ .]

**9.** The side  $BC$  of a triangle  $ABC$  is produced to  $D$  so that the triangles  $ABD$ ,  $ACD$  are equiangular. Prove that  $AD$  touches the circum-circle of the triangle  $ABC$ .

**10.** A quadrilateral  $ABCD$  is inscribed in a circle, and  $AD$ ,  $BC$  are produced to meet at  $E$ . Shew that the circle described about the triangle  $ECD$  will have the tangent at  $E$  parallel to  $AB$ .

**11.** If  $DE$  be drawn parallel to the base  $BC$  of a triangle  $ABC$  to meet the sides in  $D$  and  $E$ , shew that the circles described about the triangles  $ABC$  and  $ADE$  have a common tangent.

**12.** The diagonals of a given quadrilateral  $ABCD$  intersect at  $O$ . Shew that the centres of the circles described about the triangles  $OAB$ ,  $OBC$ ,  $OCD$ ,  $ODA$  are at the angular points of a parallelogram.

[Each of its sides is perpendicular to one of the diagonals of  $ABCD$ .]

**13.** The opposite sides of a quadrilateral inscribed in a circle are produced to meet in  $P$  and  $Q$ , and about the triangles so formed without the quadrilateral circles are described, which meet in  $R$ . Prove that  $PRQ$  is a straight line.

**14.** Three circles whose centres are  $A$ ,  $B$ , and  $C$  touch one another externally in  $D$ ,  $E$ , and  $F$ . Prove that the in-circle of the triangle  $ABC$  is the circum-circle of the triangle  $DEF$ .

[Draw the common tangents at  $D$ ,  $E$  to meet in  $O$ . Then shew that  $OFA$  is a right  $\angle$  so that the common tangent at  $F$  goes through  $O$ . Then prove that  $OA = OB = OC$  and  $OD = OE = OF$ .]

**\*\*15.** The four circles each of which passes through the centres of three of the four circles touching the sides of a triangle are equal to one another.

**16.** If  $L$ ,  $M$ ,  $N$  be any three points on the sides  $BC$ ,  $CA$ ,  $AB$  of a triangle, prove that the circles which circumscribe the triangles  $MAN$ ,  $NBL$ ,  $LCM$  meet in a point.

## PROPOSITION 6. PROBLEM.

*To inscribe a square in a given circle.*

Let ABCD be the given circle :

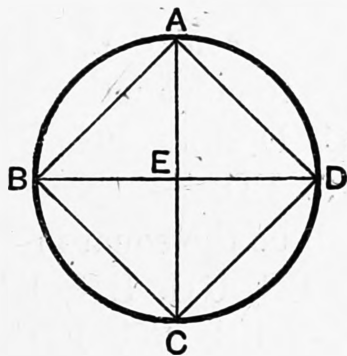
*is required to inscribe a square in ABCD.*

**Construction.** Find the centre E of the circle, and draw two diameters AC, BD at right angles to one another ;

[III. 1, I. 11.]

and join AB, BC, CD, DA.

The figure ABCD shall be the square required.



**Proof.** In the two triangles BEA, DEA,

because  $\left\{ \begin{array}{l} BE = DE, \text{ being radii,} \\ \text{and } AE \text{ is common,} \\ \text{and the right } \angle BEA = \text{the right } \angle DEA, \end{array} \right.$

$\therefore$  the base BA = the base DA.

[I. 4.]

Similarly, BC, DC each = BA, or DA ;

$\therefore$  the figure ABCD is equilateral.

Also, BD being a diameter of the circle ABCD,

BAD is a semicircle ;

[Constr.]

$\therefore$  the  $\angle BAD$  is a right  $\angle$ .

[III. 31.]

$\therefore$  the figure ABCD is equilateral, and has one angle a right angle ;

$\therefore$  it is a square.

Wherefore a square has been inscribed in the given circle.

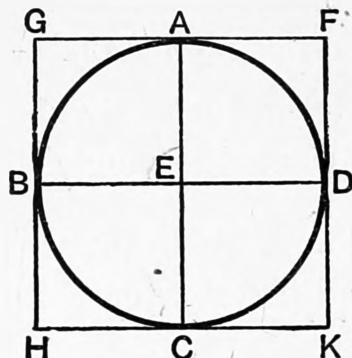
[Q.E.F.]

## PROPOSITION 7. PROBLEM.

*To describe a square about a given circle.*

Let  $ABCD$  be the given circle :

*it is required to describe a square about it.*



**Construction.** Find the centre  $E$ , and draw two diameters  $AC$ ,  $BD$  at right angles to one another ; [III. 1, I. 11.  
and through  $A$ ,  $B$ ,  $C$ ,  $D$  draw  $FG$ ,  $GH$ ,  $HK$ ,  $KF$  perpendicular to  $EA$ ,  $EB$ ,  $EC$ ,  $ED$ .

The figure  $GHKF$  shall be the square required.

**Proof.** (1) Because  $FG$ ,  $GH$ ,  $HK$ ,  $KF$  are drawn at right angles to the radii  $EA$ ,  $EB$ ,  $EC$ ,  $ED$  at their extremities, therefore they touch the circle, [III. 16, *Cor.* 1.

and the circle is inscribed in the figure  $GHKF$ .

Also because  $AEB$  and  $EBG$  are both right angles, [Constr.

$\therefore GH$  is parallel to  $AC$ . [I. 28.

Similarly,  $AC$  is parallel to  $FK$ ,

and  $GF$ ,  $HK$  are each parallel to  $BD$  ;

$\therefore$  the figures  $GK$ ,  $GC$ ,  $CF$ ,  $FB$ ,  $BK$  are parallelograms ;

$\therefore GH$  and  $FK$  each =  $AC$ ,

and  $GF$  and  $HK$  each =  $BD$ . [I. 34.

But  $AC = BD$ , both being diameters ;

$\therefore GH$ ,  $HK$ ,  $KF$ ,  $FG$  are all equal, and  $FGHK$  is equilateral.

(2) Again, since  $AEBG$  is a parallelogram, and  $AEB$  a right  $\angle$ ,

$\therefore$  the opposite  $\angle AGB$  is also a right  $\angle$ . [I. 34.

$\therefore$  the figure  $FGHK$  is equilateral, and has one of its angles a right angle ; therefore it is a square.

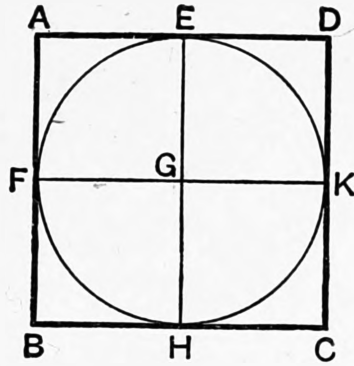
Wherefore a square has been described about the given circle. [Q.E.F.

## PROPOSITION 8. PROBLEM.

*To inscribe a circle in a given square.*

Let ABCD be the given square :

*it is required to inscribe a circle in ABCD.*



**Construction.** Bisect each of the sides AB, AD at the points F, E ; [I. 10.]

through E draw EH parallel to AB or DC, and through F draw FK parallel to AD or BC. [I. 31.]

**Proof.** Let EH and FK meet in G.

AB and AD are equal, being sides of a square ;

$\therefore$  their halves, AF and AE, are equal ; [Axiom 7.]

therefore, since AEGF is a parallelogram by construction, the opposite sides GF, GE are equal. [I. 34.]

Similarly, it may be shewn that GK = GE, and GH = GF.

$\therefore$  GE, GF, GH, GK are all equal, and the circle described with centre G, and radius equal to any one of them, will pass through the extremities of the other three ;

and it will touch AB, BC, CD, DA, because these are straight lines drawn through E, F, H, K perpendicular to the radii.

[III. 16, Cor. 1.]

Wherefore a circle has been inscribed in the given square. [Q.E.F.]

**EXERCISE.**

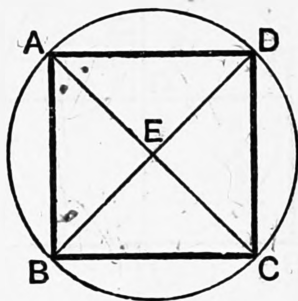
Prove the following alternative construction: Draw the diagonals AC and BD, and let them meet in O ; draw OP, OQ, OR, OS perpendicular to the sides ; the required circle has its centre at O and its radius equal to either of the four OP, OQ, OR, OS.

## PROPOSITION 9. PROBLEM.

*To describe a circle about a given square.*

Let  $ABCD$  be the given square :

*it is required to describe a circle about  $ABCD$ .*



**Construction.** Join  $AC$ ,  $BD$ , cutting one another at  $E$ .

**Proof.** In the triangles  $BAC$ ,  $DAC$ ,

because  $\begin{cases} AB = AD, \\ \text{and } AC \text{ is common,} \\ \text{and the base } BC = \text{the base } DC; \end{cases}$

$\therefore$  the  $\angle BAC =$  the  $\angle DAC$ , [I. 8.]

that is, the  $\angle BAD$  is bisected by  $AC$ .

Similarly, the other angles of the square are bisected by  $BD$  or  $AC$ .

Then, because the angles  $DAB$ ,  $ABC$  are equal,  
 $\therefore$  their halves, the angles  $EAB$ ,  $EBA$ , are equal,  
 and therefore the sides  $EB$  and  $EA$  are equal. [I. 6.]

Similarly, it may be shewn that  $EC = EB$ , and  $ED = EA$ .  
 $\therefore$   $EA$ ,  $EB$ ,  $EC$ ,  $ED$  are all equal, and the circle described with centre  $E$ , and radius equal to any one of them, will pass through the ends of the other three, and will be described about the square  $ABCD$ .

*Wherefore a circle has been described about the given square.*

[Q.E.F.]

## EXERCISES.

**1.** Describe a circle about a given rectangle.

**\*\*2.** The square inscribed in a circle is double of the square on the radius.

**\*\*3.** The square circumscribed about a circle is double of the square inscribed in the same circle.

**\*\*4.** Shew that no rectangle except a square can be described about a circle.

**5.** Describe a square about a given rectangle.

[Let ABCD be the rectangle ; at A draw a straight line outside the square making an  $\angle$  equal to half a right  $\angle$  with AD ; similarly at B, C, D ; the figure obtained is the required square.]

**6.** Inscribe a regular octagon in a given circle.

**7.** The area of a regular octagon inscribed in a circle is equal to the rectangle contained by the sides of the squares inscribed in and circumscribed about the circle.

[Let ABCD be the inscribed square and E, F, G, H the middle points of the arcs AB, BC, CD, DA. Let the tangents at A, B meet in K and AB meet OK in L. The rect. contained by the sides of the inscribed and circumscribed squares = rect. by 2 AK and AB = four times rect. AK, AL = four times rect. AL, OB = four times rect. AL, OE = eight times  $\triangle AOE$  (I. 41) = area of octagon AEBFCGDK.]

**8.** If from any point in the circumference of a given circle straight lines be drawn to the four angular points of an inscribed square, the sum of the squares on the four straight lines is double the square on the diameter. [Use Ex. 1, p. 109.]

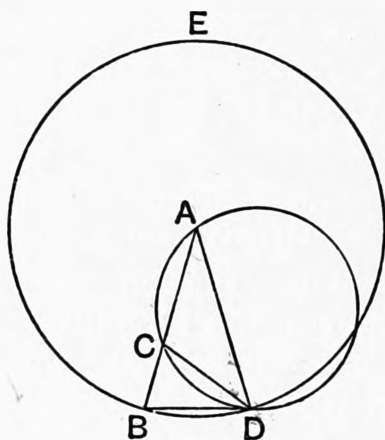
## PROPOSITION 10. PROBLEM

*To describe an isosceles triangle, having each of the angles at the base double of the third angle.*

**Construction.** Take any straight line AB, and divide it at C, so that the rect. AB, BC = the square on AC; [II. 11. with centre A and radius AB describe the circle BDE, and in it draw the chord BD equal to AC; [IV. 1. join DA.

The triangle ABD shall be such as is required.

Join DC; and about the triangle ACD describe the circle ACD. [IV. 5.



**Proof.** Because the rect. AB, BC = the square on AC, [Const. that is, = the square on BD, [Construction.  $\therefore$  BD touches the circle ACD. [III. 37.

Also, DC is drawn from the point of contact D;  $\therefore$  the  $\angle BDC$  = the  $\angle DAC$  in the alternate segment. [III. 32.

To each of these add the  $\angle CDA$ ;

$\therefore$  the whole angle BDA = the two angles CDA, DAC. [Ax. 2.

But the exterior  $\angle BCD$  = the angles CDA, DAC; [I. 32.

$\therefore$  the  $\angle BDA$  = the  $\angle BCD$ . [Axiom 1.

But the  $\angle BDA$  = the  $\angle DBA$ , since AD = AB; [I. 5.

$\therefore$  the  $\angle DBA$  also = the  $\angle BCD$ ; [Axiom 1.

$\therefore$  DC = DB; [I. 6.

but DB was made equal to CA;  $\therefore$  CA = CD, [Axiom 1.

and  $\therefore$  the  $\angle CAD$  = the  $\angle CDA$ . [I. 5.

But the angle  $BCD =$  the sum of the angles  $CAD, CDA$ ; [I. 32.

$\therefore$  the  $\angle BCD$  is double of the  $\angle CAD$ .

And the angle  $BCD$  has been shewn to be equal to each of the angles  $BDA, DBA$ ;

$\therefore$  each of the angles  $BDA, DBA$  is double of the  $\angle BAD$ .

Wherefore *an isosceles triangle has been described, having each of the angles at the base double of the third angle.* [Q.E.F.]

### EXERCISES.

**\*\*1.** Prove that the angle  $BAD$  is one-fifth part of two right angles.

**\*\*2.** Divide a right angle into 5 equal parts.

**3.** Divide a circle into two parts so that the angle in one segment may be four times that in the other.

**4.** Shew that the angle  $ACD$  in the figure of IV. 10 is equal to three times the angle at the vertex of the triangle.

**5.** Shew that the smaller of the two circles employed in the figure of IV. 10 is equal to the circle described round the required triangle.

[If two  $\triangle^s$  have equal bases and equal vertical angles then, as in the converse of III. 21, their circum-circles are equal. Also, in the above figure, we have  $BD=AC$  and  $\angle BAD=\angle ADC$ ;  $\therefore$  etc.]

**6.** Shew that in the figure of IV. 10 there are two triangles which possess the required property. Shew that there is also an isosceles triangle whose equal angles are each one-third part of the third angle.

[These two  $\triangle^s$  are  $BCD$  and  $ACD$ .]

In the figure of IV. 10, if the two circles meet again in  $F$ , prove that

**\*\*7.**  $AC$  is the side of a regular decagon inscribed in the larger circle.

**\*\*8.**  $BD$  „ „ pentagon „ „ smaller circle.

**9.**  $DF = BD$ .

[ $\angle AFD = 2$  rt.  $\angle^s - \angle ACD = \angle BCD = \angle ABD$ , etc.]

**10.**  $BF$  is the side of a regular pentagon inscribed in the larger circle.

[ $\angle FAD = \angle BAD$ ;  $\therefore \angle FAB = 2 \angle BAD =$  one fifth of four rt.  $\angle^s$ ;  $\therefore$  etc.]

**11.** If  $AF$  meet  $BD$  in  $K$ , then  $CDKF$  is a parallelogram.

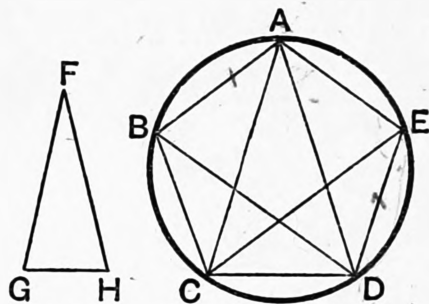


## PROPOSITION 11. PROBLEM.

*To inscribe a regular pentagon in a given circle.*

Let  $ABCDE$  be the given circle :

*it is required to inscribe a regular pentagon in the circle  $ABCDE$ .*



**Construction.** Describe an isosceles triangle  $FGH$ , having each of the angles at  $G$ ,  $H$  double of the angle at  $F$ ; [IV. 10. in the circle  $ABCDE$  inscribe the triangle  $ACD$  equiangular to the triangle  $FGH$ , [IV. 2. so that each of the angles  $ACD$ ,  $ADC$  is double of the angle  $CAD$ ; bisect the angles  $ACD$ ,  $ADC$  by the straight lines  $CE$ ,  $DB$ , [I. 9. and join  $AB$ ,  $BC$ ,  $AE$ ,  $ED$ ; then  $ABCDE$  shall be the pentagon required.

**Proof.** (1) Because each of the angles  $ACD$ ,  $ADC$  is double of the angle  $CAD$ , and that they are bisected by the straight lines  $CE$ ,  $DB$ ;  $\therefore$  the five angles  $ADB$ ,  $BDC$ ,  $CAD$ ,  $DCE$ ,  $ECA$  are equal. But equal angles stand on equal arcs;  $\therefore$  the five arcs  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$  are equal. [III. 26. Also equal arcs are subtended by equal chords; [III. 29.  $\therefore$  the five chords  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$  are equal; and therefore the pentagon  $ABCDE$  is equilateral.

(2) It is also equiangular.

For the arc  $AB =$  the arc  $DE$ ; to each add the arc  $BCD$ ;  $\therefore$  the whole arc  $ABCD =$  the whole arc  $BCDE$ . [Axiom 2.

Also the  $\angle AED$  stands on the arc  $ABCD$ , and the  $\angle BAE$  on the arc  $BCDE$ .

$\therefore$  the  $\angle AED = \text{the } \angle BAE$ . [III. 27.]

Similarly, each of the angles  $ABC$ ,  $BCD$ ,  $CDE$  is equal to the angle  $AED$  or  $BAE$ ;

$\therefore$  the pentagon  $ABCDE$  is equiangular.

Also it has been shewn to be equilateral.

Wherefore *a regular pentagon has been inscribed in the given circle.* [Q.E.F.]

### EXERCISES.

**\*\*1.** What is the magnitude of the angle of a regular pentagon?

**2.** If the alternate sides of a regular pentagon be produced to meet, the five points of intersection form another regular pentagon.

$ABCDE$  is a regular pentagon; prove that

**\*\*3.** Any angle of it is trisected by the straight lines joining it to the opposite angular points.

**4.**  $CD$  and  $AE$  meet at an angle equal to  $CAD$ .

[If  $CD$ ,  $AE$  meet in  $X$ , the  $\angle XDE = \angle CAE$  (III. 22)  $= 2 \angle CAD = \angle ACD$  and so  $\angle XED = \angle ADC$ ;  $\therefore \angle EXC = \angle CAD$ .]

**\*\*5.** The diagonals  $AC$  and  $AD$  are parallel to the sides  $ED$  and  $BC$  respectively.

**\*\*6.** All the diagonals intersect so as to form another regular pentagon.

**7.** If  $AC$  and  $BE$  meet in  $F$ , then  $AB$ ,  $CF$ , and  $EF$  are equal, and hence that any two diagonals divide one another in medial section.

[ $\angle FEC = \angle FCE = \angle$  subtended by a side;  $\therefore FE = FC$ ;  
 $\angle CFB = \angle FBA + \angle FAB = \text{twice } \angle$  subtended by a side  $= \angle CBE$ ;  
 $\therefore CF = CB = AB$ . Also  $\triangle ACE$  is a  $\triangle$  equiangular with  $\triangle ABD$  in IV. 10 and since  $\angle AEB = \angle BEC$ ,  $EF$  bisects  $\angle E$  at its base;  $\therefore$  as in IV. 10, rect.  $CA \cdot AF = CF^2$ .]

**8.**  $CFED$  is a rhombus.

**9.**  $AB$  is a tangent to the circum-circle of the triangle  $BFC$ .

**10.**  $AE$  and  $BC$  are tangents to the circum-circle of the triangle  $CFE$ .

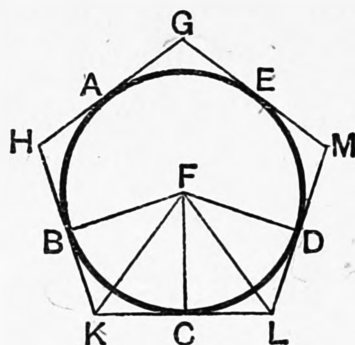
**11.**  $ABCDE$  is a regular pentagon inscribed in a circle; the arcs  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ , and  $EA$  are bisected, and lines drawn parallel to the sides of the inscribed pentagon. Prove that these lines form a circumscribing pentagon.

## PROPOSITION 12. PROBLEM.

*To describe a regular pentagon about a given circle.*

Let  $ABCDE$  be the given circle :

*it is required to describe a regular pentagon about it.*



**Construction.** Let the angular points of a regular pentagon, inscribed in the circle,\* be  $A, B, C, D, E$ , so that the arcs  $AB, BC, CD, DE, EA$  are equal ;

and through the points  $A, B, C, D, E$  draw  $GH, HK, KL, LM, MG$  touching the circle. [III. 17.]

The figure  $GHKLM$  shall be the pentagon required.

**Proof.** (1) In the triangles  $FBK, FCK$ ,  
 because  $\begin{cases} FB = FC, \text{ both being radii,} \\ \text{and } FK \text{ is common,} \\ \text{and the base } BK = \text{the base } KC, \text{ both being tangents.} \end{cases}$  [III. 17, Cor. 2.]

$\therefore$  the  $\angle BFK =$  the  $\angle CFK$ , [I. 8.]

and the  $\angle BKF =$  the  $\angle CKF$ ; [I. 4.]

$\therefore$  the  $\angle BFC =$  twice the  $\angle CFK$ ,  
 and the  $\angle BKC =$  twice the  $\angle CKF$ .

Similarly, the  $\angle CFD =$  twice the  $\angle CFL$ ,  
 and the  $\angle CLD =$  twice the  $\angle CLF$ .

And because the arc  $BC =$  the arc  $CD$ ,

the  $\angle BFC =$  the  $\angle CFD$ ; [III. 27.]

and the  $\angle BFC$  is double of the  $\angle CFK$ , and the  $\angle CFD$  is double of the  $\angle CFL$ ;

$\therefore$  the  $\angle CFK =$  the  $\angle CFL$ . [Axiom 7.]

Then, in the triangles FCK, FCL,

because  $\left\{ \begin{array}{l} \text{the } \angle CFK = \text{the } \angle CFL, \\ \text{and the right } \angle FCK = \text{the right } \angle FCL, \\ \text{and the side FC is common;} \end{array} \right.$  [Proved.

$\therefore CK = CL$ , and the  $\angle FKC = \text{the } \angle FLC$ . [I. 26.

Also because CK is equal to CL, LK is double of CK.

Similarly, it may be shewn that HK is double of BK.

And because BK is equal to CK, as was shewn, and that HK is double of BK, and LK double of CK ;

$\therefore HK = LK$ . [Axiom 6.

Similarly, it may be shewn that any two consecutive sides of the pentagon are equal, and it is therefore equilateral.

(2) It is also equiangular.

For since the  $\angle FKC = \text{the } \angle FLC$ , and that the  $\angle HKL$  is double of the  $\angle FKC$ , and the  $\angle KLM$  double of the  $\angle FLC$ , as was shewn ;

$\therefore \text{the } \angle HKL = \text{the } \angle KLM$ . [Axiom 6.

In the same manner it may be shewn that any two consecutive angles of the pentagon are equal, and it is therefore equiangular.

Also it has been shewn to be equilateral.

Wherefore a *regular pentagon has been described about the given circle*. [Q.E.F.

## EXERCISES.

**\*\*1.** Prove the following alternative construction: *In the circle inscribe, by IV. 11, a regular polygon ABCDE; draw the tangents to the circle at A, B, C, D, E, and let them meet in P, Q, R, S, T; then PQRST is the required figure.*

**\*\*2.** The bisectors of all the angles of a regular polygon meet in a point.

## PROPOSITION 13. PROBLEM.

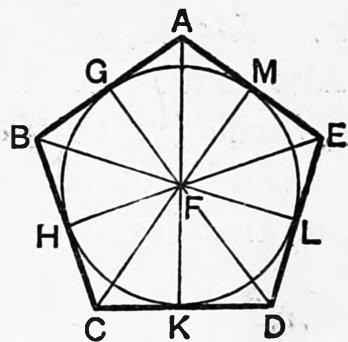
*To inscribe a circle in a given regular pentagon.*

Let  $ABCDE$  be the given regular pentagon :  
*it is required to inscribe a circle in it.*

**Construction.** Bisect the angles  $BCD$ ,  
 $CDE$  by  $CF$ ,  $DF$ ; [I. 9.]

and from the point  $F$ , at which they meet, draw the straight lines  $FB$ ,  $FA$ ,  $FE$ .

Draw  $FG$ ,  $FH$ ,  $FK$ ,  $FL$ ,  $FM$  perpendiculars to  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ . [I. 12.]



**Proof.** In the triangles  $BCF$ ,  $DCF$ ,

because  $\left\{ \begin{array}{l} BC = CD, \\ \text{and } CF \text{ is common,} \\ \text{and the } \angle BCF = \text{the } \angle DCF; \end{array} \right.$

[Hypothesis.]

$\therefore$  the base  $BF =$  the base  $DF$ ,

[Construction.]

and the  $\angle CBF =$  the  $\angle CDF$ ,

[I. 4.]

$\therefore$  twice the  $\angle CBF =$  twice the  $\angle CDF$ .

that is,  $=$  the  $\angle CDE$ , which  $=$  the  $\angle CBA$ .

that is, the  $\angle ABC$  is bisected by  $BF$ .

In the same manner it may be shewn that the angles  $BAE$ ,  $AED$  are bisected by  $AF$ ,  $EF$ .

Again, in the triangles  $FHC$ ,  $FKC$ ,

because  $\left\{ \begin{array}{l} \text{the } \angle FCH = \text{the } \angle FCK, \\ \text{and the right } \angle FHC = \text{the right } \angle FKC, \\ \text{and the side } FC \text{ is common;} \end{array} \right.$

$\therefore FH = FK$ .

[I. 26.]

Similarly, it may be shewn that  $FL$ ,  $FM$ ,  $FG$  are each equal to  $FH$  or  $FK$ ;

$\therefore FG$ ,  $FH$ ,  $FK$ ,  $FL$ ,  $FM$  are all equal, and the circle described, with centre  $F$  and radius equal to any one of them, will pass through the extremities of the other four;

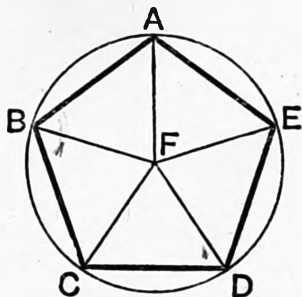
and it will touch  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ , because the angles at  $G$ ,  $H$ ,  $K$ ,  $L$ ,  $M$  are right angles. [III. 16.]

Wherefore a circle  $GHKLM$  has been inscribed in the given regular pentagon  $ABCDE$ . [Q.E.F.]

## PROPOSITION 14. PROBLEM.

*To describe a circle about a given regular pentagon.*

Let  $ABCDE$  be the given regular pentagon :  
*it is required to describe a circle about it.*



**Construction.** Bisect the angles  $BCD$ ,  $CDE$  by the straight lines  $CF$ ,  $DF$ , which meet in  $F$ . [I. 9.  
 Join  $FB$ ,  $FA$ ,  $FE$ .

**Proof.** It may be shewn, as in the preceding proposition, that the angles  $CBA$ ,  $BAE$ ,  $AED$  are bisected by the straight lines  $BF$ ,  $AF$ ,  $EF$ .

Also because the  $\angle BCD = \text{the } \angle CDE$ ,  
 and that the  $\angle FCD$  is half of the  $\angle BCD$ ,  
 and the  $\angle FDC$  is half of the  $\angle CDE$ ,

$\therefore$  the  $\angle FCD = \text{the } \angle FDC$ ; [Axiom 7.

$\therefore FC = FD$ . [I. 6.

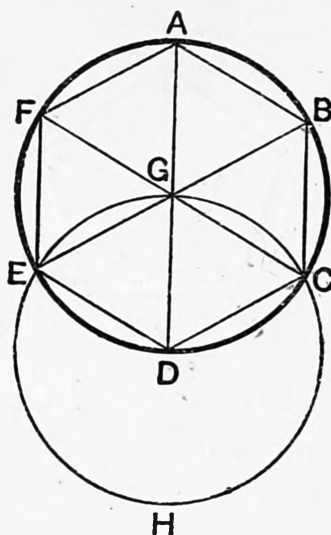
In the same manner it may be shewn that  $FB$ ,  $FA$ ,  $FE$  are each equal to  $FC$  or  $FD$ ;  
 therefore  $FA$ ,  $FB$ ,  $FC$ ,  $FD$ ,  $FE$  are all equal, and the circle described, with centre  $F$  and radius equal to any one of them, will pass through the extremities of the other four, and will be described about the pentagon  $ABCDE$ .

Wherefore a circle has been described about the given regular pentagon. [Q.E.F.

## PROPOSITION 15. PROBLEM.

*To inscribe a regular hexagon in a given circle.*

Let  $ABCDEF$  be the given circle: it is required to inscribe a regular hexagon in it.



**Construction.** Find the centre  $G$  of the circle, [III. 1.  
and draw the diameter  $AGD$ ;  
with centre  $D$  and radius  $DG$  describe the circle  $EGCH$ ,  
join  $EG$ ,  $CG$ , and produce them to meet the circumference of  
the circle  $ABCDEF$  in  $B$  and  $F$ ;  
join  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ ,  $FA$ .  
Then  $ABCDEF$  is the hexagon required.

**Proof.** (1) Because

$GE = GD$ , being radii of the circle  $ABCDEF$ ,  
and  $DE = DG$ , being radii of the circle  $EGCH$ ,

$\therefore$  the triangle  $EGD$  is equilateral; [Axiom 1.

$\therefore$  the angles  $EGD$ ,  $GDE$ ,  $DEG$  are all equal. [I. 5, Corollary.

$\therefore$  the angle  $EGD =$  one-third of two right angles. [I. 32.

Similarly, it may be shewn that the angle  $DGC =$  one-third of two right angles.

$\therefore$  the angle  $EGC =$  two-thirds of two right angles;

but the angles  $EGC$ ,  $CGB$  together  $=$  two right angles; [I. 13.

$\therefore$  the angle  $CGB =$  one-third of two right angles ;

$\therefore$  the angles  $EGD, DGC, CGB$  are equal.

Also to these are equal the vertical opposite angles  $BGA, AGF, FGE$  ; [I. 15.

$\therefore$  the six angles  $EGD, DGC, CGB, BGA, AGF, FGE$  are all equal ;

$\therefore$  the six arcs  $AB, BC, CD, DE, EF, FA$  are all equal. [III. 26.

$\therefore$  the six chords  $AB, BC, CD, DE, EF, FA$  are all equal, and the hexagon is equilateral. [III. 29.

(2) It is also equiangular.

For the arc  $AF =$  the arc  $ED$  ; to each add the arc  $ABCD$  ;

$\therefore$  the whole arc  $FABCD =$  the whole arc  $ABCDE$  ;

therefore the angles  $FED, AFE$ , which stand on these equal arcs, are equal. [III. 27.

Similarly, it may be shewn that any other two angles of the hexagon  $ABCDEF$  are equal ;

$\therefore$  the hexagon is equiangular, and it has been shewn to be equilateral, and it is inscribed in the circle  $ABCDEF$ .

Wherefore *a regular hexagon has been inscribed in the given circle.* [Q. E. F.

**Corollaries.** (1) Since  $EDG$  is an equilateral triangle, therefore  $DE = DG$ ,

that is, the side of the hexagon  $=$  the radius of the circle.

(2) If through the points  $A, B, C, D, E, F$  there be drawn tangents to the circle, a regular hexagon will be described about the circle, as may be shewn from what was said of the pentagon ; and a circle may be inscribed in a given regular hexagon, and circumscribed about it, by a method like that used for the pentagon.

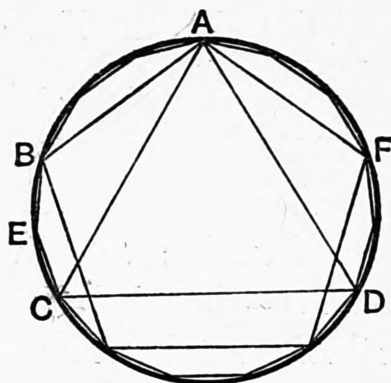


## PROPOSITION 16. PROBLEM.

*To inscribe a regular quindecagon in a given circle.*

Let ABCD be the given circle :

*it is required to inscribe a regular quindecagon in it.*



Let AC be the side of an equilateral  $\triangle$  inscribed in the circle, [IV. 2.]

and AB the side of a regular pentagon inscribed in it. [IV. 11.]

Then, of such equal parts, as the whole circumference ABCDF contains fifteen,

the arc ABC, which is the third part of the whole, contains five,

and the arc AB, which is the fifth part, contains three,

and therefore their difference, the arc BC, contains two.

Bisect the arc BC at E; [III. 30.]

therefore each of the arcs BE, EC is the fifteenth part of the whole circumference.

$\therefore$  if BE, EC be drawn, and chords equal to them be placed round in the whole circle, [IV. 1.]

a regular quindecagon will be inscribed in it. [Q.E.F.]

**Corollary.** As in the case of the pentagon, if through the points of division made by inscribing the quindecagon tangents be drawn to the circle, a regular quindecagon will be described about it; and also, as for the pentagon, a circle may be inscribed in a given regular quindecagon, and circumscribed about it.

## EXERCISES.

**1.** AB, BC are sides of a regular hexagon, and therefore AC is the side of an equilateral triangle inscribed in the same circle. Prove that the square on AC is three times the square on AB.

**2.** In the figure of IV. 15, prove that A, C, E are the angular points of an equilateral triangle.

**\*\*3.** The area of a regular hexagon is twice that of an equilateral triangle inscribed in the same circle.

[In the figure of IV. 15, AGBC is a  $\parallel^{\text{gm}}$ , and  $\therefore = \text{twice } \triangle AGC$ . But the hexagon = three times AGCB and  $\triangle AEC = \text{three times } \triangle AGC$ , etc.]

**4.** Construct on a given straight line as one side a regular—

(1) pentagon ; (2) hexagon ; (3) octagon.

**\*\*5.** Inscribe a figure of 30 sides in a given circle.

**\*\*6.** Any equilateral figure which is inscribed in a circle is also equiangular.

**7.** If the alternate sides of a regular polygon be produced to meet, the points of intersection form another regular polygon of the same number of sides.

**\*\*8.** Prove that the centre of the inscribed circle and of the circumscribing circle of any regular polygon is found by bisecting any two consecutive angles.

**\*\*9.** If we have any regular polygon in a circle and draw tangents to the circle at its vertices, the latter form a regular circumscribed polygon having the same number of the sides as the first.

**10.** Verify that, by the constructions of this book, we can inscribe in a circle polygons of respectively  $2 \times 2^n$ ,  $3 \times 2^n$ ,  $5 \times 2^n$ , and  $15 \times 2^n$  sides, where  $n$  is any positive integer.

[See also Notes, page 328 ]

## BOOK V.

### DEFINITIONS.

1. A less magnitude is said to be an aliquot part, or a **measure**, or **submultiple**, of a greater magnitude, when the less is contained a certain number of times exactly in the greater.

2. A greater magnitude is said to be a **multiple** of a less, when the greater contains the less a certain number of times exactly.

3. Ratio is a mutual relation of two magnitudes of the same kind to one another in respect of quantity

The two magnitudes are called the terms of the ratio. The first term is called the **antecedent**, and the second term the **consequent**.

4. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

[This definition amounts to saying that the quantities must be of the *same kind*.]

5. The first of four magnitudes is said to have the same **ratio** to the second that the third has to the fourth, when any equimultiples whatever of the first and the third being taken, and any equimultiples whatever of the second and the fourth; if according as the multiple of the first is greater than, equal to, or less than the second, the multiple of the third is greater than, equal to, or less than the fourth.

[Thus, to use the language of Algebra, four magnitudes, A, B, C, D, are proportionals, if  $mC \cong nD$  according as  $mA \cong mB$ , where  $m$  and  $n$  are any whole numbers whatever.]

[*Note.* The fifth definition is the foundation of Euclid's doctrine of proportion. The student will find in works on Algebra (for example, in the present Editors' *Algebra for Beginners*, Arts. 373-377) a comparison of Euclid's definition of proportion with the simpler definitions which are employed in Arithmetic and Algebra. Euclid's definition is applicable to *incommensurable* quantities, as well as to *commensurable* quantities.

We should recommend the student to read the first proposition of the sixth Book immediately after the fifth definition of the fifth Book ; he will there see how Euclid applies his definition, and will thus obtain a better notion of its meaning and importance.]

6. Magnitudes which have the same ratio are called **proportionals**.

When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second as the third is to the fourth. The first and fourth magnitudes are sometimes called the **extremes**, and the second and third the **means**.

7. When of the equimultiples of four magnitudes, taken as in the fifth definition, the multiple of the first is greater than the multiple of the second, but the multiple of the third is not greater than the multiple of the fourth, then the first is said to have to the second a greater ratio than the third has to the fourth ; and the third is said to have to the fourth a less ratio than the first has to the second.

8. Analogy, or **proportion**, is the similitude of ratios.

9. Proportion consists in three terms at least.

10. Three, or more, magnitudes are said to be in **continued proportion** when the ratio of the first to the second is the same as that of the second to the third, and the ratio of the second to the third is that of the third to the fourth, and so on.

When three magnitudes are continued proportionals, the first is said to have to the third the **duplicate ratio** of that which it has to the second.

Thus, if the ratio of A to B equals that of B to C, then A has to C the duplicate ratio of that which it has to B.

[The second magnitude is said to be a **mean proportional** between the first and the third.]

[It will be shewn in the corollary to VI. 20 that the duplicate ratio of two straight lines is the same as the ratio of the squares on the lines.]

When four magnitudes are continued proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second.

**11.** *Definition of compound ratio.* When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio which is compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio that E has to F; and B to C the same ratio that G has to H; and C to D the same ratio that K has to L; then, by this definition, A is said to have to D the ratio compounded of the ratios of E to F, G to H, and K to L.

[Since the ratio of A to C is compounded of that of A to B and of B to C, it follows from the preceding definition that the ratio compounded of two equal ratios is the duplicate of either of them.]

**12.** In proportionals, the antecedent terms are said to be **homologous** to one another: as also the consequents to one another.

Geometers make use of the following technical words to signify certain ways of changing either the order or the magnitude of four proportionals, so that they continue still to be proportionals.

**13.** *Permutando*, or *alternando*, by permutation or alternately; when it is inferred that the first is to the third as the second is to the fourth. V. 16.

**14.** *Invertendo*, by inversion ; when it is inferred that the second is to the first as the fourth is to the third. V. B.

**15.** *Componendo*, by composition ; when it is inferred that the first, together with the second, is to the second as the third, together with the fourth, is to the fourth. V. 18.

**16.** *Dividendo*, by division ; when it is inferred that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth. V. 17.

**17.** *Convertendo*, by conversion ; when it is inferred that the first is to its excess above the second as the third is to its excess above the fourth. V. E.

**18.** *Ex æquali distantia*, or *ex æquo*, by equality ; when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each set, and it is inferred that the first is to the last of the first set of magnitudes as the first is to the last of the others.

Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken.

**19.** *Ex æquali*, by direct equality. This term is used simply by itself, when the first magnitude is to the second of the first set as the first is to the second of the other set ; and the second is to the third of the first set as the second is to the third of the other ; and so on in order ; and the inference is that mentioned in the preceding definition. V. 22.

Thus, suppose the magnitudes of the first set to be A, B, C, ..., and those of the second set to be P, Q, R, ..., and suppose we know that  
                                           A is to B as P to Q,  
 and                                        B to C as Q to R ;  
 then we infer that           A is to C as P to R.

**20.** *Ex æquali in proportione perturbatâ seu inordinatâ*, from equality in perturbate or disorderly proportion, that is, by transverse equality. This term is used when the first magnitude is to the second of the first set as the last but one is

to the last of the second set ; and the second is to the third of the first set as the last but two is to the last but one of the second set ; and the third is to the fourth of the first set as the last but three is to the last but two of the second set ; and so on in a cross order ; and the inference is that mentioned in the eighteenth definition. V. 23.

Thus, if	A is to B as Q to R,
and	B is to C as P to Q,
we infer that	A is to C as P to R.

### AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

2. Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

*Note 1.* In the proofs of the propositions of this Book, capital letters A, B, C, ... will be used to denote the magnitudes themselves and small letters  $m$ ,  $n$ ,  $p$ , ... to denote whole numbers.

Thus,  $mA$  will mean the magnitude obtained by adding together  $m$  magnitudes each equal to A.

The symbol  $>$  will sometimes be used for *greater than*, and the symbol  $<$  for *less than*.

The notation  $A : B$  is often used as an abbreviation for "the ratio of A to B." Also the notation  $A : B :: C : D$ , or  $A : B = C : D$ , is often used as an abbreviation for "the ratio of A to B equals that of C to D," which, again, is generally abbreviated into "A is to B as C is to D."

*Note 2.* The propositions of the Fifth Book might be divided into four sections. Propositions 1 to 6 relate to the properties of equimultiples. Propositions 7 to 10 and 13 and 14 connect the notion of the *ratio* of magnitudes with the ordinary notions of *greater*, *equal*, and



*less.* Propositions 11, 12, 15, and 16 may be considered as introduced to shew that, *if four quantities of the same kind be proportionals they will also be proportionals when taken alternately.* The remaining propositions shew that magnitudes are proportional by *composition*, by *division*, and *ex æquo*.

In this division of the Fifth Book propositions 13 and 14 are supposed to be placed immediately after proposition 10; and they might be taken in this order without any change in Euclid's demonstrations.

The propositions headed *A, B, C, D, E*, and also the preceding axioms, were supplied by Simson.

*Note 3.* At the end of each proposition has been appended the corresponding algebraic theorem; many of these theorems are so simple that they are self-evident. The others are proved in the *Algebra for Beginners*, new edition, Arts. 357-364 and 371.

### PROPOSITION 1. THEOREM.

*If any number of magnitudes be equimultiples of as many, each of each; whatever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.*

Let any number of magnitudes *A, B, C* be equimultiples of as many others *E, F, G*, each of each:  
*whatever multiple A is of E, the same multiple shall A, B, and C together be of E, F, and G together.*

For let *A* contain *E*, *B* contain *F*, and *C* contain *G* any number of times, say *m* times.

Then  $A = mE$ ,  $B = mF$ , and  $C = mG$ ;

$\therefore A, B, \text{ and } C \text{ together} = mE, mF, \text{ and } mG \text{ together}$   
 $= m \text{ times } E, F, \text{ and } G \text{ together.}$

Wherefore, *if any number, etc.*

[Q. E. D.]

**Algebraically.** If  $a = me$ ,  $b = mf$ , and  $c = mg$ ,  
then  $a + b + c = m(e + f + g).$

### PROPOSITION 2. THEOREM.

*If the first be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; the first, together with the fifth, shall be the*



*same multiple of the second that the third, together with the sixth, is of the fourth.*

Let the first magnitude A be the same multiple of B the second that C the third is of D the fourth, and let E the fifth be the same multiple of B the second that F the sixth is of D the fourth :

*then A, together with E, shall be the same multiple of B that C, together with F, is of D.*

For let  $m$  be the first common multiple, so that

$$A = mB, \quad C = mD.$$

Also, let  $n$  be the second common multiple, so that

$$E = nB, \quad F = nD;$$

$\therefore$  A, together with E,  $= mB$ , together with  $nB$ ,  $= (m + n)B$ ,  
and C, together with F,  $= mD$ , together with  $nD$ ,  $= (m + n)D$ .

**Corollary.** If  $A = mB$ ,  $C = mD$ ,

$$E = nB, \quad F = nD,$$

and

$$G = pB, \quad H = pD,$$

then A, E, G together are the same multiple of B that C, F, H together are of D.

Wherefore, *if the first, etc.*

[Q. E. D.]

**Algebraically.** If  $a = mb$ ,  $c = md$ ,  $e = nb$ , and  $f = nd$ ,

then

$$\frac{a + e}{b} = \frac{c + f}{d}.$$

### PROPOSITION 3. THEOREM.

*If the first be the same multiple of the second that the third is of the fourth, and if of the first and the third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.*

Let A the first be the same multiple of B the second that C the third is of D the fourth ; and of A and C let the equimultiples  $nA$ ,  $nC$  be taken :

*$nA$  shall be the same multiple of B that  $nC$  is of D,*

For let  $A$  be  $m$  times  $B$ , and  $C$   $m$  times  $D$ , so that

$$A = mB, \text{ and } C = mD.$$

Then  $n$  times  $A = n$  times  $mB = mn$  times  $B$ ,

and  $n$  times  $C = n$  times  $mD = mn$  times  $D$ ;

$\therefore nA$  is the same multiple of  $B$  that  $nC$  is of  $D$ .

Wherefore, *if the first, etc.*

[Q. E. D.]

**Algebraically.** If  $a = mb$  and  $c = md$ ,

then 
$$\frac{na}{b} = \frac{nc}{d}.$$

#### PROPOSITION 4. THEOREM.

*If the first have the same ratio to the second that the third has to the fourth, and if there be taken any equimultiples whatever of the first and the third, and also any equimultiples whatever of the second and the fourth, then the multiple of the first shall have the same ratio to the multiple of the second that the multiple of the third has to the multiple of the fourth.*

Let  $A$  the first have to  $B$  the second the same ratio that  $C$  the third has to  $D$  the fourth; and of  $A$  and  $C$  let there be taken any equimultiples whatever  $mA$  and  $mC$ , and of  $B$  and  $D$  any equimultiples whatever  $nB$  and  $nD$ :

$mA$  shall have the same ratio to  $nB$  that  $mC$  has to  $nD$ .

For of  $mA$  and  $mC$  take any equimultiple  $p$ ,

and of  $nB$  and  $nD$  take any equimultiple  $q$ ;

then  $mA$  has to  $nB$  the same ratio that  $mC$  has to  $nD$ ,

if  $pmA$  is greater, equal to, or less than  $qnB$ ,  
according as  $pmC$  is greater, equal to, or less than  $qnD$ ,  
that is, if  $A$  has to  $B$  the same ratio that  $C$  has to  $D$ ,

[V. Definition 5.]

since  $pmA$ ,  $pmC$  are equimultiples of  $A$  and  $C$ ,

and  $qnB$ ,  $qnD$  are equimultiples of  $B$  and  $D$ .

Wherefore, *if the first, etc.*

[Q. E. D.]

**Algebraically.** If  $a : b :: c : d$ , then  $ma : nb :: mc : nd$ .

## PROPOSITION 5. THEOREM.

*If one magnitude be the same multiple of another that a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder that the whole is of the whole.*

Let A be the same multiple of B that C, taken from the first, is of D taken from the other :  
*the remainder E shall be the same multiple of the remainder F that the whole A is of the whole B.*

For let  $A = mB$  and  $C = mD$ .

Then the difference between A and C

= the difference between  $mB$  and  $mD$ ,

that is,  $E =$  the difference between  $mB$  and  $mD$ ,

that is,  $= m$  times the difference between B and D,

that is,  $E = m$  times F.

Wherefore, *if one magnitude, etc.*

[Q. E. D.]

**Algebraically.** If  $a = mb$  and  $c = md$ , then  $a - c = m(b - d)$ .

## PROPOSITION 6. THEOREM.

*If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainders shall be either equal to these others, or equimultiples of them.*

Let the two magnitudes A, B be equimultiples of the two C, D ; and let E, F, taken from the first two, be equimultiples of the same C, D :

*the remainders G, H shall be equimultiples of C and D.*

For let  $A = mC$  and  $B = mD$ ,

and let  $E = nC$  and  $F = nD$ .

Then the difference between A and E

= the difference between  $mC$  and  $nC$ ,

that is,  $G = (m - n)$  times C ;

so  $H = (m - n)$  times D.

Wherefore, *if two magnitudes, etc.*

[Q. E. D.]

**Algebraically.** If  $a = mc$ ,  $b = md$ ,  $e = nc$ , and  $f = nd$ ,  
 then 
$$\frac{a - e}{c} = \frac{b - f}{d}.$$

PROPOSITION A. THEOREM.

*If the first of four magnitudes have the same ratio to the second that the third has to the fourth, then, if the first be greater than the second, the third shall also be greater than the fourth ; and if equal, equal ; and if less, less.*

The four magnitudes being A, B, C, and D, we know that if we take any equimultiples whatever  $mA$ ,  $mC$ ,  $nB$ , and  $nD$ , then as  $mA$  is greater, equal to, or less than  $nB$ ,

so  $mC$  is greater, equal to, or less than  $nD$  ; [V. Def. 5.  
 $\therefore$ , if we make  $m$  and  $n$  each equal to unity, it follows that

as A is greater, equal to, or less than B,

so C is greater, equal to, or less than D.

Wherefore, *if the first, etc.*

[Q.E.D.]

**Algebraically.** If  $a : b :: c : d$ , then  $a \cong b$  according as  $c \cong d$ .

PROPOSITION B. THEOREM.

*If four magnitudes be proportionals, they shall also be proportionals when taken inversely.*

Let A be to B as C is to D :

then also, inversely, B shall be to A as D is to C.

Take  $mA$ ,  $mC$ , any equimultiples of A and C, and  $nB$ ,  $nD$ , any equimultiples of B and D.

Then we are given that if  $mA$  is greater than  $nB$ ,

then  $mC$  is greater than  $nD$ ,

that is, if  $nB$  is less than  $mA$ , then  $nD$  is less than  $mC$ .

Similarly, if  $nB$  is equal to, or greater than  $mA$ , then  $nD$  is equal to, or greater than  $mC$  ;

$\therefore nB$  is less than, equal to, or greater than  $mA$ , according as  $nD$  is less than, equal to, or greater than  $mC$ ;

$\therefore B$  is to  $A$  as  $D$  to  $C$ .

[V. Definition 5.]

Wherefore, *if four magnitudes, etc.*

[Q.E.D.]

**Algebraically.** If  $a : b :: c : d$ , then  $b : a :: d : c$ .

### PROPOSITION C. THEOREM.

*If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first shall be to the second as the third is to the fourth.*

Let  $A, B, C, D$  be the four magnitudes,  
and, firstly, let  $A$  and  $C$  be equimultiples of  $B$  and  $D$ ,  
so that  $A = mB$  and  $C = mD$ ,  
then shall  $A$  be to  $B$  as  $C$  to  $D$ .

Take any multiples  $nA, nC$  of  $A$  and  $C$ ,  
and any multiples  $pB, pD$  of  $B$  and  $D$ .  
Then  $A$  is to  $B$  as  $C$  to  $D$ ,

if  $nC$  is greater than, equal to, or less than  $pD$ ,  
according as  $nA$  is greater than, equal to, or less than  $pB$ ;

that is, if  $nmD$  is greater than, equal to, or less than  $pD$ ,  
according as  $nmB$  is greater than, equal to, or less than  $pB$ ;

that is, if  $nm$  is greater than, equal to, or less than  $p$ ,  
according as  $nm$  is greater than, equal to, or less than  $p$ ,  
which is true.

Next, let  $A$  be the same part, or submultiple, of  $B$  that  $C$  is of  $D$ .

Then  $B$  is the same multiple of  $A$  that  $D$  is of  $C$ ;

$\therefore$  by the first part,  $B$  is to  $A$  as  $D$  to  $C$ ;

$\therefore$  inversely,  $A$  is to  $B$  as  $C$  to  $D$ .

[V. B.]

Wherefore, *if the first, etc.*

[Q.E.D.]

**Algebraically.** If  $a = mb$  and  $c = md$ , then  $a : b :: c : d$ .

## PROPOSITION D. THEOREM.

*If the first be to the second as the third is to the fourth, and if the first be a multiple, or a part, of the second, the third shall be the same multiple, or the same part, of the fourth.*

*First*, let  $A = mB$ , and take the same equimultiple  $mD$  of  $D$ ; then shall  $C = mD$ .

Since  $A, B, C, D$  are proportionals,

$\therefore$  according as  $A$  is greater, equal to, or less than  $mB$ ,  
so  $C$  is greater, equal to, or less than  $mD$ .

But  $A = mB$ ;  $\therefore C = mD$ .

*Next*, let  $A$  be a part of  $B$ .

Because  $A$  is to  $B$  as  $C$  to  $D$ ,  
 $\therefore$  inversely,  $B$  is to  $A$  as  $D$  to  $C$ ;  
 $\therefore B$  is the same multiple of  $A$  that  $D$  is of  $C$ , by the first case;

that is,  $A$  is the same part of  $B$  that  $C$  is of  $D$ .

Wherefore, *if the first be, etc.*

[Q.E.D.]

**Algebraically.** If  $a : b :: c : d$ , and if  $a = mb$ , then  $c = md$ .

## PROPOSITION 7. THEOREM.

*Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.*

Let  $A$  and  $B$  be equal magnitudes, and  $C$  any other magnitude:

*each of the magnitudes  $A$  and  $B$  shall have the same ratio to  $C$ ; and  $C$  shall have the same ratio to each of the magnitudes  $A$  and  $B$ .*

For it is clear, since  $A = B$ ,

that  $mA$  is greater than, equal to, or less than  $nC$ ,  
according as  $mB$  is greater than, equal to, or less than  $nC$ ;

$\therefore A$  is to  $C$  as  $B$  to  $C$ ;

$\therefore$  inversely,  $C$  is to  $A$  as  $C$  to  $B$ .

[V. B.]

Wherefore, *equal magnitudes, etc.*

[Q.E.D.]

**Algebraically.** If  $a = b$ , then  $a : c :: b : c$  and  $c : a :: c : b$ .

## PROPOSITION 8. THEOREM.

*Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.*

Let  $A$  and  $B$  be unequal magnitudes, of which  $A$  is the greater; and let  $C$  be any other magnitude whatever:  
 *$A$  shall have a greater ratio to  $C$  than  $B$  has to  $C$ ; and  $C$  shall have a greater ratio to  $B$  than it has to  $A$ .*

Let  $m$  be such a number that  $mB$  and  $m$  times  $A - B$  are both greater than  $C$ , and let  $nC$  be the least multiple of  $C$  which is just greater than  $mA$ ;

$\therefore nC - C$  is less than  $mA$ ; that is,  $mA$  is  $> (n - 1)C$ .

Again, since  $nC$  is greater than  $mA$ ,

but  $C$  is less than  $mA - mB$ ,

$\therefore (n - 1)C$  is greater than  $mB$ ;

$\therefore mA$  is greater, but  $mB$  is less, than  $(n - 1)$  times  $C$ ;

$\therefore$  the ratio of  $A$  to  $C$  is greater than that of  $B$  to  $C$ . [V. Def. 7.]

Again, since we have proved that  $(n - 1)$  times  $C$  is less than  $mA$ , but greater than  $mB$ ;

$\therefore$  the ratio of  $C$  to  $A$  is less than that of  $C$  to  $B$ . [V. Def. 7.]

Wherefore, *of unequal magnitudes, etc.*

[Q.E.D.]

**Algebraically.** If  $a > b$ , then  $a : c > b : c$ , and  $c : b < c : a$ .

## PROPOSITION 9. THEOREM.

*Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.*

First, let  $A$  and  $B$  have the same ratio to  $C$ :

*$A$  shall be equal to  $B$ .*

For if not,  $A$  must be greater than  $B$ , or else  $B$  greater than  $A$ .

$A$  is not greater than  $B$ ; for then the ratio of  $A$  to  $C$  would be greater than that of  $B$  to  $C$ . [V. 8.]

Neither is B greater than A; for then the ratio of B to C would be greater than that of A to C; [V. 8.]

$\therefore$  A must be equal to C.

In a similar manner it may be shewn that, if C have the same ratio to A or B, then A and B are equal.

Wherefore, *magnitudes which have, etc.*

[Q. E. D.]

**Algebraically.** If  $a : c :: b : c$ , then  $a = b$ .

Also if  $a = b$ , then  $a : c :: b : c$ .

### PROPOSITION 10. THEOREM.

*That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.*

First, let A have to C a greater ratio than B has to C:

A shall be greater than B.

Since the ratio of A to C is greater than that of B to C, two numbers  $m$  and  $n$  can be found, such that  $mA$  is greater than  $nC$ , and  $mB$  is less than  $nC$ ; [V. Definition 7.]

$\therefore mA$  is greater than  $mB$ ;  $\therefore$  A is greater than B.

Next, if the ratio of C to B is greater than that of C to A, two numbers  $p$  and  $q$  can be found, such that  $pC$  is greater than  $qB$ , and  $pC$  is less than  $qA$ ;

$\therefore qA$  is greater than  $qB$ ;  $\therefore$  A is greater than B.

Wherefore, *that magnitude, etc.*

[Q. E. D.]

**Algebraically.** If  $a : c > b : c$ , then  $a > b$ ;

and if  $c : b > c : a$ , then  $b < a$ .

### PROPOSITION 11. THEOREM.

*Ratios that are equal to the same ratio are equal to one another.*

Let A be to B as C is to D, and let C be to D as E is to F:

A shall be to B as E is to F.



Take any equimultiples  $mA$ ,  $mC$ ,  $mE$ , and any equimultiples  $nB$ ,  $nD$ ,  $nF$ .

Then, by hypothesis,

$mA$  is greater than, equal to, or less than  $nB$ ,

as  $mC$  is greater than, equal to, or less than  $nD$ ,

that is, as  $mE$  is greater than, equal to, or less than  $nF$ ; [*Hyp.*

$\therefore A$  is to  $B$  as  $E$  to  $F$ .

Wherefore, *ratios that are the same, etc.*

[Q.E.D.]

**Algebraically.** If  $a:b::c:d$  and  $c:d::e:f$ , then  $a:b::e:f$ .

### PROPOSITION 12. THEOREM.

*If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents be to all the consequents.*

Let any number of magnitudes  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$  be proportionals; namely, as  $A$  is to  $B$ , so let  $C$  be to  $D$ , and  $E$  to  $F$ : as  $A$  is to  $B$ , so shall  $A$ ,  $C$ ,  $E$  together be to  $B$ ,  $D$ ,  $F$  together.

Take any equimultiples  $mA$ ,  $mC$ ,  $mE$ ,

and any equimultiples  $nB$ ,  $nD$ ,  $nF$ .

Then, by hypothesis, if  $mA$  is greater than  $nB$ ,

$mC$  is greater than  $nD$ , and  $mE$  is greater than  $nF$ ;

$\therefore mA$ ,  $mC$ ,  $mE$  are together greater than the sum of  $nB$ ,  $nD$ ,  $nF$ .

Similarly, if  $mA$  is equal to, or less than  $nB$ ,

$mA$ ,  $mC$ ,  $mE$  together are equal to, or less than the sum of  $nB$ ,  $nD$ ,  $nF$ ;

$\therefore m$  times the sum of  $A$ ,  $C$ ,  $E$  is greater than, equal to, or less than  $n$  times the sum of  $B$ ,  $D$ ,  $F$ ,

according as  $mA$  is greater, equal to, or less than  $nB$ ;

$\therefore A$ ,  $C$ ,  $E$  together are to  $B$ ,  $D$ ,  $F$  together as  $A$  to  $B$ .

Wherefore, *if any number, etc.*

[Q.E.D.]

**Algebraically.** If  $a:b::c:d::e:f$ ,

then  $a:b::a+c+e:b+d+f$ .

## PROPOSITION 13. THEOREM.

*If the first have the same ratio to the second which the third has to the fourth, but the third to the fourth a greater ratio than the fifth to the sixth, the first shall have to the second a greater ratio than the fifth has to the sixth.*

Let A have the same ratio to B that C has to D, but C a greater ratio to D than E to F :

A shall have to B a greater ratio than E has to F.

Because the ratio of C to D is greater than that of E to F, there are two numbers  $m, n$ , such that  $mC > nD$ , but  $mE < nF$ .

[V. Definition 7.

But if  $mC > nD$ , then  $mA > nB$  ;

$\therefore mA > nB$ , but  $mE < nF$  ;

$\therefore$  the ratio of A to B is greater than that of E to F.

Wherefore, *if the first, etc.*

[Q.E.D.

**Algebraically.** If  $a:b::c:d$  and  $c:d > e:f$ , then  $a:b > e:f$ .

## PROPOSITION 14. THEOREM.

*If the first have the same ratio to the second that the third has to the fourth, then if the first be greater than the third the second shall be greater than the fourth ; and if equal, equal ; and if less, less.*

Let A have the same ratio to B that C has to D :

*if A be greater than C, B shall be greater than D ; if equal, equal ; and if less, less.*

If  $A > C$ , then the ratio of A to B is greater than that of C to B,

[V. 8.

that is, the ratio of C to D is greater than that of C to B ;

[Hypothesis.

$\therefore B > D$ .

[V. 10.

Similarly, it may be shewn that if  $A = C$ , then  $B = D$ ,

and if  $A < C$ , then  $B < D$ .

Wherefore, *if the first, etc.*

[Q.E.D.

**Algebraically.** If  $a:b::c:d$ , then  $a \geq c$  according as  $b \geq d$ .

## PROPOSITION 15. THEOREM.

*Magnitudes have the same ratio to one another that their equimultiples have.*

Let A and B be any two magnitudes, and let any two equimultiples  $mA$  and  $mB$  be taken :

*the ratio of A to B shall be the same as that of  $mA$  to  $mB$ .*

Since A is to B as A to B ;

$\therefore$  A is to B as  $A + A$  to  $B + B$ ,

that is, as  $2A$  to  $2B$ .

[V. 12.]

Again, since A is to B as  $2A$  to  $2B$  ;

$\therefore$  A is to B as  $A + 2A$  to  $B + 2B$ ,

that is, as  $3A$  to  $3B$ .

[V. 12.]

Similarly, A is to B as  $4A$  to  $4B$ ,

[V. 12.]

and so on whatever be the equimultiple  $m$ .

Wherefore, *magnitudes, etc.*

[Q.E.D.]

**Algebraically.**  $a : b :: ma : mb$ .

## PROPOSITION 16. THEOREM.

*If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.*

Let A, B, C, D be four magnitudes of the same kind which are proportionals ; namely, as A is to B so let C be to D : *they shall also be proportionals when taken alternately, that is, A shall be to C as B is to D.*

Let any equimultiples  $mA$ ,  $mB$ , and any equimultiples  $nC$ ,  $nD$  be taken.

Then the ratio of  $mA$  to  $mB$  = the ratio of A to B [V. 15.]

= the ratio of C to D

[Hypothesis.]

= the ratio of  $nC$  to  $nD$  ;

[V. 15.]

$\therefore mA$ ,  $mB$ ,  $nC$ ,  $nD$  are proportionals ;

$\therefore mA$  is greater than, equal to, or less than  $nC$ , according as  $mB$  is greater than, equal to, or less than  $nD$  ;

[V. 14.]

$\therefore A$ ,  $C$ ,  $B$ ,  $D$  are proportionals.

[V. Definition 5.]

Wherefore, *if four magnitudes, etc.*

[Q.E.D.]

**Algebraically.** If  $a : b :: c : d$ , then  $a : c :: b : d$ .

## PROPOSITION 17. THEOREM.

*If four magnitudes be proportional, the difference between the first and second shall be to the second as the difference between the third and fourth is to the fourth.*

Let  $A, B, C, D$  be four magnitudes, such that  $A$  is to  $B$  as  $C$  to  $D$ :

*the difference between  $A, B$  shall be to  $B$  as the difference between  $C, D$  is to  $D$ .*

Take any equimultiples  $mA, mC$ , and any equimultiples  $nB, nD$ , and let  $p$  denote the difference between  $n$  and  $m$ .

First, let  $mA > nB$ ; then, by hypothesis,  $mC > nD$ .

Since  $mA > nB$ ,  $\therefore mA - mB > nB - mB$ ,

that is,  $m(A - B) > pB$ .

Similarly  $m(C - D) > pD$ ;

$\therefore m(A - B) > pB$ , when  $m(C - D) > pD$ ;

so  $m(A - B) \leq pB$ , when  $n(C - D) \leq pD$ ;

$\therefore A - B, B, C - D$ , and  $D$  are proportionals. [V. Definition 5.]

Wherefore, *if four magnitudes, etc.*

[Q. E. D.]

**Algebraically.** If  $a : b :: c : d$ , then  $a - b : b :: c - d : d$ .

## PROPOSITION 18. THEOREM.

*If four magnitudes be proportionals, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

Let  $A, B, C, D$  be four magnitudes, such that  $A$  is to  $B$  as  $C$  to  $D$ :

*then shall the sum of  $A$  and  $B$  be to  $B$  as the sum of  $C$  and  $D$  to  $D$ .*

Take any equimultiples  $mA, mC$ , and any equimultiples  $nB, nD$ , and let  $p$  denote the sum of  $m$  and  $n$ .

First, let  $mA > nB$ ; then, by hypothesis,  $mC > nD$ .

Since  $mA > nB$ ;  $\therefore mA + mB > mB + nB$ ,

that is,  $m(A + B) > pB$ .

So since  $mC > nD$ ,  $\therefore m(C + D) > pD$ ;

$\therefore$  if  $m(A + B) > pB$ , then  $m(C + D) > pD$ .

Similarly, it can be shewn that

if  $m(A + B) \leq pB$ , then  $m(C + D) \leq pD$ ;

$\therefore A + B, B, C + D, D$  are proportionals.

Wherefore, *if four magnitudes, etc.*

[Q.E.D.]

**Algebraically.** If  $a : b :: c : d$ , then  $a + b : b :: c + d : d$ .

### PROPOSITION 19. THEOREM.

*If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole is to the whole.*

Let  $A$  be to  $B$  as  $C$  to  $D$ :

then shall  $A - C$  be to  $B - D$  as  $A$  to  $B$ .

For since  $A$  is to  $B$  as  $C$  to  $D$ ;

$\therefore A$  is to  $C$  as  $B$  to  $D$ ;

[V. 16.]

$\therefore A - C$  is to  $C$  as  $B - D$  to  $D$ ;

[V. 17.]

$\therefore A - C$  is to  $B - D$  as  $C$  to  $D$ ,

[V. 16.]

that is, as  $A$  to  $B$ .

Wherefore, *if a whole, etc.*

[Q.E.D.]

**Algebraically.** If  $a : b :: c : d$ , then  $a - c : b - d :: a : b$ .

### PROPOSITION E. THEOREM.

*If four magnitudes be proportionals, they shall also be proportionals by conversion; that is, the first shall be to its excess above the second as the third is to its excess above the fourth.*

Let  $A$  be to  $B$  as  $C$  is to  $D$ :

then shall  $A$  be to  $A - B$  as  $C$  to  $C - D$ .

Since  $A$  is to  $B$  as  $C$  to  $D$ ;

$\therefore A$  is to  $C$  as  $B$  to  $D$ ;

[V. 16.]

$\therefore A - B$  is to  $C - D$  as  $A$  to  $C$ ;

[V. 19.]

$\therefore A - B$  is to  $A$  as  $C - D$  to  $C$ ;

[V. 16.]

$\therefore A$  is to  $A - B$  as  $C$  to  $C - D$ . [V. B.]

Wherefore, if four magnitudes, etc. [Q.E.D.]

**Algebraically.** If  $a : b :: c : d$ , then  $a : a - b :: c : c - d$ .

### PROPOSITION 20. THEOREM.

*If there be three magnitudes, and other three, which have the same ratio, taken two and two, then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.*

Let  $A, B, C$  be three magnitudes, and  $D, E, F$  other three, such that  $A$  is to  $B$  as  $D$  is to  $E$ , and  $B$  to  $C$  as  $E$  to  $F$ : if  $A$  be greater than  $C$ ,  $D$  shall be greater than  $F$ ; and if equal, equal; and if less, less.

*First*, let  $A$  be greater than  $C$ ,  
then  $A$  has to  $B$  a greater ratio than  $C$  has to  $B$ . [V. 8.]

But  $A$  is to  $B$  as  $D$  is to  $E$ ; [Hypothesis.]

$\therefore D$  has to  $E$  a greater ratio than  $C$  has to  $B$ . [V. 13.]

Also because  $B$  is to  $C$  as  $E$  is to  $F$ , [Hypothesis.]

$\therefore$  by inversion,  $C$  is to  $B$  as  $F$  is to  $E$ ; [V. B.]

$\therefore D$  has to  $E$  a greater ratio than  $F$  has to  $E$ ;

$\therefore D$  is greater than  $F$ . [V. 10.]

*Secondly*, let  $A = C$ ;  $\therefore A$  is to  $B$  as  $C$  is to  $B$ . [V. 7.]

But  $A$  is to  $B$  as  $D$  is to  $E$ , [Hypothesis.]

and  $C$  is to  $B$  as  $F$  is to  $E$ ; [Hypothesis, V. B.]

$\therefore D$  is to  $E$  as  $F$  is to  $E$ ; [V. 11.]

$\therefore D = F$ . [V. 9.]

*Lastly*, let  $A < C$ , then  $C > A$ ;

$\therefore$  as in the first case,  $C$  is to  $B$  as  $F$  is to  $E$ ,

and  $B$  is to  $A$  as  $E$  is to  $D$ ;

therefore, by the first case,  $F > D$ , that is,  $D < F$ .

Wherefore, if there be three, etc. [Q.E.D.]

**Algebraically.** If  $a : b :: d : e$  and  $b : c :: e : f$ , then  $a \geq c$  according as  $d \geq f$ .

## PROPOSITION 21. THEOREM.

*If there be three magnitudes, and other three, which have the same ratio, taken two and two, but in a cross order ; then, if the first be greater than the third, the fourth shall be greater than the sixth ; and if equal, equal ; and if less, less.*

Let A, B, C be three magnitudes, and D, E, F other three, such that A is to B as E is to F, and B to C as D to E : if A be greater than C, D shall be greater than F ; and if equal, equal ; and if less, less.

*First*, let  $A > C$  ;

$\therefore$  A has to B a greater ratio than C has to B. [V. 8.]

But A is to B as E to F ; [Hypothesis.]

$\therefore$  E has to F a greater ratio than C has to B. [V. 13.]

Also because B is to C as D to E, [Hypothesis.]

$\therefore$  by inversion, C is to B as E to D ; [V. B.]

$\therefore$  E has to F a greater ratio than E has to D ;

$\therefore F < D$ , that is,  $D > F$ . [V. 10.]

*Secondly*, let  $A = C$  ;  $\therefore$  A is to B as C to B. [V. 7.]

But A is to B as E to F, [Hypothesis.]

and C is to B as E to D ; [Hypothesis, V. B.]

$\therefore$  E is to F as E to D, [V. 11.]

and therefore  $D = F$ . [V. 9.]

*Lastly*, let  $A < C$ , then  $C > A$  ;

$\therefore$  as in the first case, C is to B as E to D,  
and B is to A as F to E ;

$\therefore$  by the first case,  $F > D$ , that is,  $D < F$ .

Wherefore, *if there be three, etc.* [Q.E.D.]

**Algebraically.** If  $a : b :: e : f$  and  $b : c :: d : e$ , then  $a \geq c$  according as  $d \geq f$ .

## PROPOSITION 22. THEOREM.

*If there be any number of magnitudes, and as many others, which have the same ratio, taken two and two in order, the first shall have*

*to the last of the first magnitudes the same ratio which the first of the others has to the last.*

[This proposition is usually cited by the words *ex æquali*.]

*First*, let there be three magnitudes A, B, C, and other three D, E, F, such that A is to B as D is to E, and B to C as E to F:

A shall be to C as D is to F.

Of A and D take any equimultiples whatever  $mA$ ,  $mD$ ; of B and E any equimultiples  $nB$ ,  $nE$ ; and of C and F any equimultiples  $pC$ ,  $pF$ .

Because A is to B as D to E;

[*Hypothesis*.

$\therefore mA$  is to  $nB$  as  $mD$  to  $nE$ .

[V. 4.

Similarly  $nB$  is to  $pC$  as  $nE$  to  $pF$ ;

[V. 4.

$\therefore$  according as  $mA$  is greater, equal to, or less than  $pC$ ,

so is  $mD$  greater, equal to, or less than  $pF$ ;

[V. 20.

$\therefore$  A is to C as D to F.

[V. *Definition* 5.

*Next*, let there be four magnitudes A, B, C, D, and other four E, F, G, H, such that A is to B as E to F, B to C as F to G, and C to D as G to H

*then shall A be to D as E to H.*

For, by the first case, A is to C as E to G.

But C is to D as G to H;

[*Hypothesis*.

$\therefore$  by the first case, A is to D as E to H.

A similar demonstration applies whatever be the number of magnitudes.

Wherefore, *if there be any number, etc.*

[Q. E. D.

**Algebraically.** If  $a:b::d:e$  and  $b:c::e:f$ , then  $a:c::d:f$ .

### PROPOSITION 23. THEOREM.

*If there be any number of magnitudes, and as many others, which have the same ratio, taken two and two in a cross order, the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.*



*First*, let there be three magnitudes, A, B, C, and other three E, F, G, such that A is to B as F to G, and B to C as E to F :

*A shall be to C as E to G.*

Of A and E take any equimultiples whatever  $mA$ ,  $mE$  ;  
 of B and F any equimultiples  $nB$ ,  $nF$  ;  
 and of C and G any equimultiples  $nC$ ,  $nG$ .

Because A is to B as F to G ; [Hypothesis.

$\therefore mA$  is to  $mB$  as  $nF$  to  $nG$ . [V. 15 and 11.

Also since B is to C as E to F ; [Hypothesis.

$\therefore mB$  is to  $nC$  as  $mE$  to  $nF$  ; [V. 16, 15, 11.

$\therefore$  according as  $mA$  is greater, equal to, or less than  $nC$ , so is  $mE$  greater, equal to, or less than  $nG$  ; [V. 21.

$\therefore$  A is to C as E to G. [V. Definition 5.

*Next*, let there be four magnitudes, A, B, C, D, and other four E, F, G, H, such that A is to B as G to H, B to C as F to G, and C to D as E to F :

*then A is to D as E to H.*

For, by the first case, A is to C as F to H.

But C is to D as E to F ; [Hypothesis.

$\therefore$  A is to D as E to H, by the first case.

Similarly, whatever be the number of the magnitudes.

Wherefore, *if there be any number, etc.* [Q. E. D.

**Algebraically.** If  $a : b :: f : g$  and  $b : c :: e : f$ ,  
 then  $a : c :: e : g$ .

*Note.* Propositions 22 and 23 may be included in one enunciation, thus : *Ratios which are compounded of equal ratios are equal.*

#### PROPOSITION 24. THEOREM.

*If the first have to the second the same ratio which the third has to the fourth, and the fifth have to the second the same ratio which the sixth has to the fourth, then the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.*

Let A be to B as C to D, and E to B as F to D:  
*then shall A and E together be to B as C and F together to D.*

Because E is to B as F to D;

$\therefore$  B is to E as D to F, by inversion. [V. B.

But A is to B as C to D; [Hypothesis.

$\therefore$  *ex æquali*, A is to E as C to F; [V. 22.

$\therefore$  A + E is to E as C + F to F. [V. 18.

But E is to B as F to D; [Hypothesis.

$\therefore$  *ex æquali*, A + E is to B as C + F to D. [V. 22.

Wherefore, *if the first have, etc.* [Q. E. D.

**Algebraically.** If  $a : b :: c : d$  and  $e : b :: f : d$ ,  
 then  $a + e : b :: c + f : d$ .

#### PROPOSITION 25. THEOREM.

*If four magnitudes of the same kind be proportionals, the greatest and least of them together shall be greater than the other two together.*

Let the four magnitudes A, B, C, D be proportionals;  
 namely, let A be to B as C to D; and let A be the greatest  
 of them, and consequently D the least: [V. A, V. 14.

*A and D together shall be greater than B and C together.*

Since A is to B as C to D;

$\therefore$  A is to C as B to D, *alternando*; [V. 16.

$\therefore$  A is to A - C as B is to B - D. [V. E.

But A is greater than B; [Hypothesis.

$\therefore$  A - C is greater than B - D; [V. 14.

$\therefore$  A + D is greater than B + C.

Wherefore, *if four magnitudes, etc.* [Q. E. D.

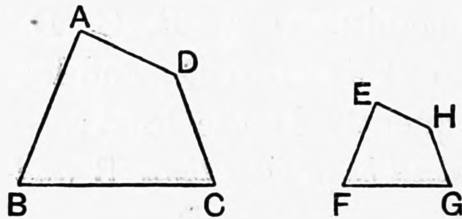
**Algebraically.** If  $a : b :: c : d$ , and  $a$  be the greatest of the  
 four quantities, then  $a + d > b + c$ .

## BOOK VI.

### DEFINITIONS.

1. Two rectilineal figures are said to be **equiangular** when the angles of the first, taken in order, are respectively equal to the angles of the second, taken in order.

2. **Similar** rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals. Also the corresponding sides of the two figures are said to be **homologous**.



Thus, in the case of the two figures  $ABCD$ ,  $EFGH$ , they will be similar if the angles at  $A$ ,  $B$ ,  $C$ ,  $D$  are respectively equal to the angles  $E$ ,  $F$ ,  $G$ ,  $H$ ; and if

$AB$  is to  $BC$  as  $EF$  to  $FG$ ,

$BC$  is to  $CD$  as  $FG$  to  $GH$ ,

$CD$  is to  $DA$  as  $GH$  to  $HE$ ;

and

and  $\therefore$  by Book V.,  $DA$  to  $AB$  as  $HE$  to  $EF$ .

Also  $AB$  and  $EF$  are homologous; also  $BC$  and  $FG$ ,  $CD$  and  $GH$ , etc.

3. **Reciprocal** figures, namely, triangles and parallelograms, have their sides about two of their angles proportionals in such a manner, that a side of the first figure is to a side of the other, as the remaining side of this other is to the remaining side of the first.

Thus, if  $ABC$ ,  $DEF$  be two triangles which have their sides about the angles  $A$  and  $D$  such that  $AB$  is to  $DE$  as  $DF$  to  $AC$ , they are reciprocal figures.

4. A straight line is said to be cut in **extreme and mean ratio** when the whole is to the greater segment as the greater segment is to the less.

Thus, in II. 11, the straight line  $AB$  is divided at  $H$  in extreme and mean ratio.

5. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.



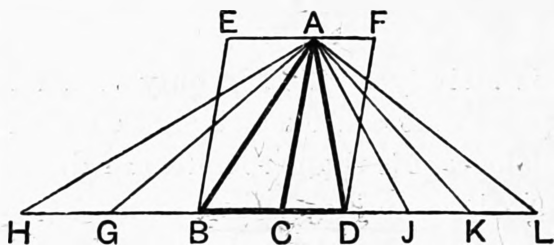
[This definition is strictly applicable only to a triangle because no other figure has a point that can exclusively be called its vertex. The altitude of a parallelogram is the perpendicular drawn to its base from any point in the opposite side.]

## PROPOSITION 1. THEOREM.

*Triangles and parallelograms of the same altitude are to one another as their bases.*

Let the triangles  $ABC$ ,  $ACD$ , and the parallelograms  $EC$ ,  $CF$  have the same altitude, namely, the perpendicular drawn from  $A$  to  $BD$ :

*as the base  $BC$  is to the base  $CD$ , so shall the triangle  $ABC$  be to the triangle  $ACD$ , and the parallelogram  $EC$  to the parallelogram  $CF$ .*



**Construction.** Produce  $BD$  both ways ;  
take any number of straight lines  $BG$ ,  $GH$ , each equal to  $BC$ ,  
and any number  $DJ$ ,  $JK$ ,  $KL$ , each equal to  $CD$ , [I. 3.  
and join  $AG$ ,  $AH$ ,  $AJ$ ,  $AK$ ,  $AL$ .

**Proof.** (1) Because  $CB$ ,  $BG$ ,  $GH$  are all equal, [*Construction*.  
the triangles  $ABC$ ,  $AGB$ ,  $AHG$  are all equal ; [I. 38.  
 $\therefore$  the  $\triangle AHC$  is the same multiple of the  $\triangle ABC$  that  $HC$   
is of  $BC$ .

Similarly, the  $\triangle ACL$  is the same multiple of the  $\triangle ACD$   
that  $CL$  is of  $CD$ .

And if  $HC = CL$ , the  $\triangle AHC =$  the  $\triangle ACL$  ;  
and if  $HC$  be greater than  $CL$ , the  $\triangle AHC$  is greater than  
the  $\triangle ACL$  ; and if less, then less. [I. 38.

Then, since there are four magnitudes, namely, the two  
triangles  $ABC$ ,  $ACD$ , and the two bases  $BC$ ,  $CD$  ;  
and of the first and the third any equimultiples have been  
taken, namely, the  $\triangle AHC$  and the base  $HC$  ;  
and of the second and fourth any equimultiples whatever have  
been taken, namely, the  $\triangle ACL$  and the base  $CL$  ;

and since it has been shewn that the  $\triangle AHC$  is greater, equal to, or less than the  $\triangle ACL$ , according as  $HC$  is greater, equal to, or less than  $CL$ ;

$\therefore$  the  $\triangle ABC$  : the  $\triangle ACD$  :: the base  $BC$  : the base  $CD$ .

[V. Definition 5.

(2) Because the parallelogram  $CE$  is double of the  $\triangle ABC$ , and the parallelogram  $CF$  is double of the  $\triangle ACD$ ; [I. 41. and that magnitudes have the same ratio which their equimultiples have ; [V. 15.

$\therefore$  the parallelogram  $EC$  : the parallelogram  $CF$   
:: the  $\triangle ABC$  : the  $\triangle ACD$ ,

that is, :: the base  $BC$  : the base  $CD$ . [V. 11.

Wherefore, *triangles, etc.* [Q.E.D.

**Corollary.** From this it is plain that triangles and parallelograms which have equal altitudes, are to one another as their bases.

For, let the figures be placed so as to have their bases in the same straight line, and to be on the same side of it, and let there be drawn perpendiculars from the vertices of the triangles to the bases; then the straight line which joins the vertices is parallel to that in which their bases are; [I. 33. because the perpendiculars are both equal and parallel to one another. 28.

For, if the same construction be made as in the proposition, the demonstration will be the same.

### EXERCISES.

1. The four triangles into which any quadrilateral is divided by its two diagonals are proportional.

2. Perpendiculars are drawn from any point within an equilateral triangle on the three sides: shew that their sum is always the same.

[Let  $O$  be any point within the equilateral  $\triangle ABC$ ;  $OP$ ,  $OQ$ ,  $OR$  the  $\perp^{\text{rs}}$  upon the sides;  $AD$ ,  $BE$ ,  $CF$  the  $\perp^{\text{rs}}$  from the angular points. Then we have  $\triangle OBC$  :  $\triangle ABC$  ::  $OP$  :  $AD$ ;  $\triangle OCA$  :  $\triangle ABC$  ::  $OQ$  :  $BE$ , that is, as  $OQ$  :  $AD$ , etc.;  $\therefore$  sum of  $\triangle^s OBC$ ,  $OCA$ ,  $OAB$  :  $\triangle ABC$ ;  $\therefore$  sum of  $OP$ ,  $OQ$ ,  $OR$  :  $AD$ ;  $\therefore OP$ ,  $OQ$ ,  $OR$  together =  $AD$ .]

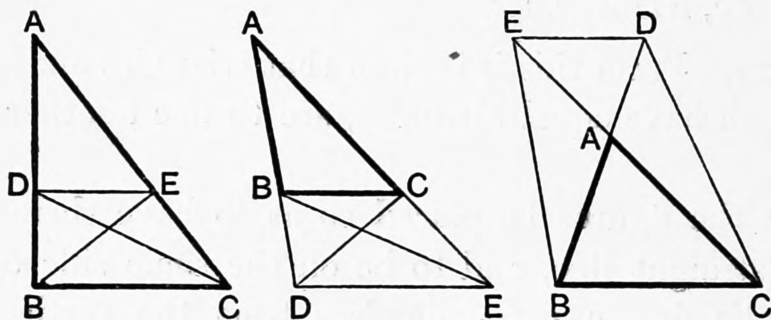
## PROPOSITION 2. THEOREM.

*If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those sides produced, proportionally.*

*Conversely, if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.*

Let DE be drawn parallel to BC, one of the sides of the triangle ABC :

BD shall be to DA as CE to EA.



**Construction.** Join BE, CD.

**Proof.** The  $\triangle BDE =$  the  $\triangle CDE$ , because they are on the same base DE and between the same parallels DE, BC. [I. 37. Also ADE is another  $\triangle$ , and equal magnitudes have the same ratio to the same magnitude ; [V. 7.

$\therefore$  the  $\triangle BDE : \text{the } \triangle ADE :: \text{the } \triangle CDE : \text{the } \triangle ADE$ .

But the  $\triangle BDE : \text{the } \triangle ADE :: BD : DA$ , because they have the same altitude, namely, the perpendicular drawn from E to AB. [VI. 1.

Similarly, the  $\triangle CDE : \text{the } \triangle ADE :: CE : EA$  ;

$\therefore BD : DA :: CE : EA$ . [V. 11.

*Conversely*, let BD be to DA as CE to EA, and join DE : DE shall be parallel to BC.

For, the same construction being made, because  $BD : DA :: CE : EA$ ,

[Hypothesis.



and  $BD : DA :: \text{the } \triangle BDE : \text{the } \triangle ADE$ , [VI. 1.

and  $CE : EA :: \text{the } \triangle CDE : \text{the } \triangle ADE$ ; [VI. 1.

$\therefore \text{the } \triangle BDE : \text{the } \triangle ADE :: \text{the } \triangle CDE : \text{the } \triangle ADE$ ; [V. 11.

$\therefore \text{the } \triangle BDE = \text{the } \triangle CDE$ . [V. 9.

Also these triangles are on the same base DE and on the same side of it; therefore DE is parallel to BC. [I. 39.

Wherefore, *if a straight line, etc.* [Q.E.D.

### EXERCISES.

1. The straight line joining the middle points of the sides of a triangle is parallel to the base and is one half the base.

\*\*2. A fixed point O is joined to all the points in a given straight line RS; prove that all the points which divide the joining lines in the same ratio lie on a straight line which is parallel to RS.

3. D is any point in the side BC of a triangle ABC; if BD, DC, BA, AC are bisected in E, F, G, H respectively, prove that EG is equal and parallel to FH.

\*\*4. If any two straight lines be cut by three parallel straight lines, the intercepts on the one are proportional to the corresponding intercepts on the other.

\*\*5. The diagonals of a trapezium cut one another proportionally, and any straight line drawn parallel to either of its parallel sides will cut the other sides in the same ratio.

6. Shew that the diagonals of a quadrilateral, two of whose sides are parallel and one of them double of the other, cut one another at a point of trisection.

7. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G: shew that FG is parallel to CD.

8. From any point in the base of a triangle straight lines are drawn parallel to the sides: shew that the intersection of the diagonals of every parallelogram so formed lies in a certain straight line.

9. In a triangle ABC a straight line AD is drawn perpendicular to the straight line BC which bisects the angle B; shew that a straight line drawn from D parallel to AC will bisect AB.

10. ABC is a triangle; it is required to draw from a given point P, in the side AB, or AB produced, a straight line to AC, or AC produced, so that it may be bisected by BC. [Bisect PA in Q; draw QR  $\parallel$  to AC to meet BC in R; PR produced is the required line.]



## PROPOSITION 3. THEOREM.

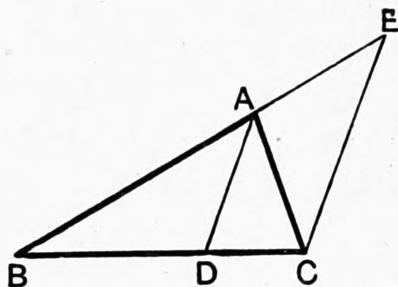
*If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another.*

*Conversely, if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.*

Let  $ABC$  be a triangle, and let the angle  $BAC$  be bisected by  $AD$ , which meets the base at  $D$  :

$BD$  shall be to  $DC$  as  $BA$  is to  $AC$ .

**Construction.** Through  $C$  draw  $CE$  parallel to  $DA$ , [I. 31. and let  $BA$  produced meet  $CE$  at  $E$ .



**Proof.** (i.) Because  $AC$  meets the parallels  $AD$ ,  $EC$ ,  
the  $\angle CAD =$  the  $\angle ACE$ , and the  $\angle BAD =$  the  $\angle AEC$ ; [I. 29.]

But the  $\angle CAD =$  the  $\angle BAD$ ; [Hypothesis.]

$\therefore$  the  $\angle ACE =$  the  $\angle AEC$ ; [Axiom 1.]

$\therefore AE = AC$ . [I. 6.]

Also, because  $AD$  is parallel to  $EC$ , [Construction.]

one of the sides of the  $\triangle BCE$ ;

$\therefore BD : DC :: BA : AE$ ; [VI. 2.]

$\therefore BD : DC :: BA : AC$ .

(ii.) *Conversely*, let  $BD : DC :: BA : AC$ , and join  $AD$  :  
*the angle BAC shall be bisected by AD.*

Let the same construction be made.

Then  $BD : DC :: BA : AC$ , [Hypothesis.

and  $BD : DC :: BA : AE$ , [VI. 2.

because  $AD$  is parallel to  $EC$ ; [Construction.

$\therefore BA : AC :: BA : AE$ ; [V. 11.

$\therefore AC = AE$ ; [V. 9.

$\therefore$  the  $\angle AEC =$  the  $\angle ACE$ . [I. 5.

But the  $\angle AEC =$  the exterior  $\angle BAD$ , [I. 29.

and the  $\angle ACE =$  the alternate  $\angle CAD$ ; [I. 29.

$\therefore$  the  $\angle BAD =$  the  $\angle CAD$ , [Axiom 1.

that is, the  $\angle BAC$  is bisected by  $AD$ .

Wherefore, *if the vertical angle, etc.*

[Q.E.D.

## EXERCISES

1. The side  $BC$  of a triangle  $ABC$  is bisected at  $D$ , and the angles  $ADB$ ,  $ADC$  are bisected by the straight lines  $DE$ ,  $DF$ , meeting  $AB$ ,  $AC$  at  $E$ ,  $F$  respectively: shew that  $EF$  is parallel to  $BC$ .

2. If the bisectors of the angles  $A$ ,  $C$  of a quadrilateral  $ABCD$  meet on the diagonal  $BD$ , prove that the bisectors of the angles  $B$ ,  $D$  meet on the diagonal  $AC$ .

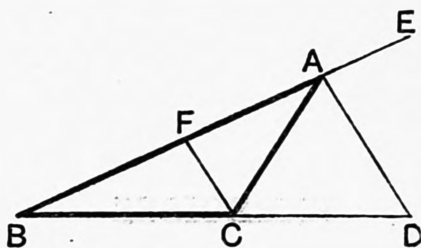
## PROPOSITION A. THEOREM.

*If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line which also cuts the base produced, the segments between the bisector and the extremities of the base shall have the same ratio which the other sides of the triangle have to one another.*

*Conversely, if the segments of the base produced have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section shall bisect the exterior angle of the triangle.*

In the triangle ABC let one of its sides BA be produced to E; and let the exterior angle CAE be bisected by AD which meets the base produced at D :

BD shall be to DC as BA is to AC.



**Construction.** Through C draw CF parallel to AD, [I. 31. meeting AB at F.

**Proof.** (i.) Because AC meets the parallels CF, AD,  
the  $\angle CAD =$  the alternate  $\angle ACF$ ,  
and the exterior  $\angle DAE =$  the interior opposite  $\angle AFC$ ; [I. 29.

But the  $\angle CAD =$  the  $\angle DAE$ ; [Construction.

$\therefore$  the  $\angle ACF =$  the  $\angle AFC$ , and  $AF = AC$ . [I. 6.

And because AD is parallel to FC, [Construction.

one of the sides of the  $\triangle BCF$ ;

$\therefore BD : DC :: BA : AF$ ; [VI. 2

but  $AF = AC$ ;

$\therefore BD : DC :: BA : AC$ .

(ii.) *Conversely*, let BD be to DC as BA to AC; and join AD:

*the exterior  $\angle$ CAE shall be bisected by AD.*

Let the same construction be made.

Then  $BD : DC :: BA : AC$ , [Hypothesis.

and  $BD : DC :: BA : AF$ ; [VI. 2.

$\therefore BA : AC :: BA : AF$ ; [V. 11.

$\therefore AC = AF$ , [V. 9.

and  $\therefore$  the  $\angle ACF =$  the  $\angle AFC$ . [I. 5.

But the  $\angle AFC =$  the exterior  $\angle DAE$ ; [I. 29.

and the  $\angle ACF =$  the alternate  $\angle CAD$ ; [I. 29.

$\therefore$  the  $\angle CAD =$  the  $\angle DAE$ , [Axiom 1.

that is, the  $\angle CAE$  is bisected by the straight line AD.

Wherefore, *if the exterior angle, etc.* [Q.E.D.

*Note.* Propositions 3 and 4 may be enunciated in one statement, thus: *If the interior, or exterior, vertical angle of a triangle be bisected by a straight line which meets the base, the segments of the base [II. Def. 3] are to one another as the sides of the triangle.*

### EXERCISES.

1. In the circumference of the circle of which AB is a diameter, take any point P; and draw PC, PD on opposite sides of AP, and equally inclined to it, meeting AB at C and D: shew that AC is to BC as AD is to BD.

2. AB is a straight line and C a point in it; find a point D in AB produced such that AD is to DB as AC to CB. [Use Ex. 1.]

3. From the same point A straight lines are drawn making the angles BAC, CAD, DAE each equal to half a right angle, and they are cut by a straight line BCDE, which makes BAE an isosceles triangle: shew that BC or DE is a mean proportional between BE and CD.

[By VI. 3,  $BD : DC :: BA : AC$ ;

$\therefore BD - DC : BD + DC :: BA - AC : BA + AC$  (V. 18, 17).

But  $BD - DC = BD + OD - OC = BO + 2 OD - OC =$  twice OD,

and  $BD + DC = BC =$  twice OB;  $\therefore$  etc.]

4. The angle A of a triangle ABC is bisected by AD which cuts the base at D, and O is the middle point of BC: shew that OD bears the same ratio to OB that the difference of the sides bears to their sum.

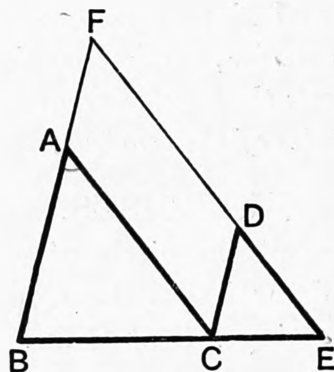
## PROPOSITION 4. THEOREM.

*The sides about the equal angles of triangles which are equiangular to one another are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or the consequents of the ratios.*

Let the triangle  $ABC$  be equiangular to the triangle  $DCE$ , having the angle  $ABC$  equal to the angle  $DCE$ , and the angle  $ACB$  equal to the angle  $DEC$ , and consequently the angle  $BAC$  equal to the angle  $CDE$ :

*the sides about the equal angles of the triangles  $ABC$ ,  $DCE$  shall be proportionals, viz.,*

$AB$  to  $BC$  as  $DC$  to  $CE$ ,  
 $BC$  to  $CA$  as  $CE$  to  $ED$ ,  
 and  $CA$  to  $AB$  as  $ED$  to  $DC$ .



**Proof.** (1) Let the  $\triangle DCE$  be placed so that  $CE$  may be contiguous to  $BC$ , and in the same straight line with it, and  $D$  and  $A$  on the same side of  $BCE$ . [I. 22.]

Because the  $\angle^s ABC$ ,  $ACB$  are together  $<$  two right  $\angle^s$ . [I. 32.]  
 and that the  $\angle ACB =$  the  $\angle CED$ ; [Hypothesis.]

$\therefore$  the angles  $ABC$ ,  $CED$  are together  $<$  two right angles;  
 $\therefore$   $BA$  and  $ED$ , if produced, will meet. [Axiom 12.]

Let them be produced and meet at the point  $F$ .

Then, because the  $\angle ABC =$  the  $\angle DCE$ , [Hypothesis.]

$BF$  is parallel to  $CD$ ; [I. 28.]

and because the  $\angle ACB =$  the  $\angle DEC$ , [Hypothesis.]

$AC$  is parallel to  $FE$ . [I. 28.]

$\therefore$   $FACD$  is a  $\parallel^m$ ;  $\therefore AF = CD$ , and  $AC = FD$ . [I. 34.]

(2) Because AC is parallel to FE, a side of the  $\triangle FBE$ ,

$$\therefore BA : AF :: BC : CE ; \quad [\text{VI. 2.}]$$

but  $AF = CD$  ;

$$\therefore BA : CD :: BC : CE ; \quad [\text{V. 7.}]$$

and, alternately,  $AB : BC :: DC : CE$ . [V. 16.]

(3) Again, because CD is parallel to BF,

$$\therefore BC : CE :: FD : DE ; \quad [\text{VI. 2.}]$$

but  $FD = AC$  ;

$$\therefore BC : CE :: AC : DE ;$$

and, alternately,  $BC : AC :: CE : DE$ . V. 16.

Also it has been shewn that  $AB : BC :: DC : CE$  ;

therefore, *ex æquali*,  $AB : AC :: DC : DE$ . [V. 22.]

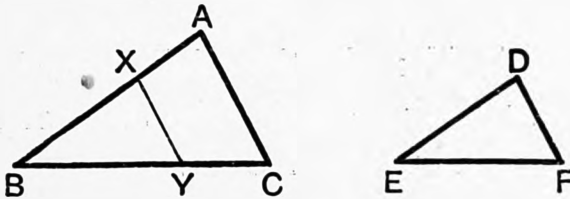
Wherefore, *the sides, etc.* [Q. E. D.]

**Corollary.** Hence *Triangles which are equiangular are also similar.* This is not necessarily true of other rectilineal figures.

#### ALTERNATIVE PROOF OF PROPOSITION 4.

Let the side AB be greater than the corresponding side DE.

Superpose the triangle DEF upon the triangle ABC, so that the angle E coincides with the angle B, and let X, Y be the points in BA, BC with which D and F coincide.



Then  $ED = BX$ ,  $EF = BY$ , and  $\angle BXY = \angle EDF$  ;

$$\therefore \angle BXY = \angle BAC ;$$

[*Hypothesis.*]

$\therefore XY$  and  $AC$  are parallel ;

$\therefore AX$  is to  $XB$  as  $CY$  to  $YB$  ;

[VI. 2.]

$\therefore$  *componendo*,  $AB : BX :: CB : BY$  ;

[V. 18.]

$\therefore$  alternately,  $AB : BC :: BX : BY$ ,

that is,  $\therefore DE : EF$ .

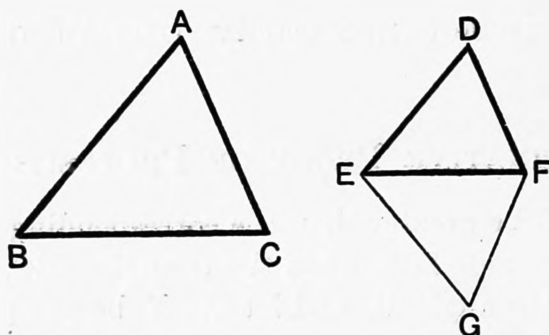
Similarly, by applying the  $\triangle DEF$  to the  $\triangle ABC$  so that the angle F coincides with C, we can shew that  $BC : CA :: EF : FD$  ;

and therefore, *ex æquali*,  $AB : AC :: DE : DF$ .

## PROPOSITION 5. THEOREM.

*If the sides of two triangles taken in order, about each of their angles, be proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to the homologous sides.*

Let the triangles  $ABC$ ,  $DEF$  have their sides proportional, so that  $AB$  is to  $BC$  as  $DE$  to  $EF$ ; and  $BC$  to  $CA$  as  $EF$  to  $FD$ ; and, consequently, *ex æquali*,  $BA$  to  $AC$  as  $ED$  to  $DF$ : *the two triangles shall be equiangular, and shall have the  $\angle ABC$  equal to the  $\angle DEF$ , the  $\angle BCA$  equal to the  $\angle EFD$ , and the  $\angle BAC$  equal to the  $\angle EDF$ .*



**Construction.** At  $E$  make the  $\angle FEG$  equal to the  $\angle ABC$ , the points  $D$  and  $G$  being on opposite sides of  $EF$ ; and at  $F$  make the  $\angle EFG$  equal to the  $\angle BCA$ . [I. 23.]

**Proof.** The remaining  $\angle EGF =$  the remaining  $\angle BAC$ ;  
 $\therefore$  the  $\triangle ABC$  is equiangular to the  $\triangle GEF$ ;

$$\therefore AB : BC :: GE : EF. \quad [\text{VI. 4.}]$$

$$\text{But } AB : BC :: DE : EF; \quad [\text{Hypothesis.}]$$

$$\therefore DE : EF :: GE : EF; \quad [\text{V. 11.}]$$

$$\therefore DE = GE.$$

$$\text{Similarly, } DF = GF.$$

Then, in the triangles DEF, GEF,

because  $\left\{ \begin{array}{l} DE = GE, \\ \text{and } EF \text{ is common,} \\ \text{and the base } DF = \text{the base } GF; \end{array} \right.$

$\therefore$  the  $\angle DEF =$  the  $\angle GEF$ ,

and the  $\angle DFE =$  the  $\angle GFE$ .

[I. 8.]

But the  $\angle GEF =$  the  $\angle ABC$ ;

[Construction.]

$\therefore$  the  $\angle ABC =$  the  $\angle DEF$ .

[Axiom 1.]

For the same reason, the  $\angle ACB =$  the  $\angle DFE$ ,

and the  $\angle$  at A = the  $\angle$  at D;

[I. 32.]

$\therefore$  the  $\triangle ABC$  is equiangular to the  $\triangle DEF$ .

Wherefore, *if the sides, etc.*

[Q. E. D.]

*Note.* Each of the two propositions VI. 4 and VI. 5 is the converse of the other. They shew that if two triangles have either of the two properties involved in the definition of similar figures they will have the other also, that is, *if two triangles are equiangular they have their sides proportional; and if they have their sides proportional they are equiangular.* This is a special property of triangles. In other figures either of the properties may exist alone. For example, any rectangle and a square have their angles equal, but not their sides proportional; while a square and any rhombus have their sides proportional, but not their angles equal.

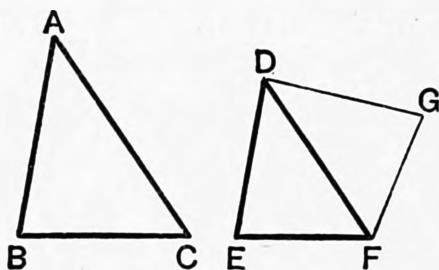
[For Exercises on Propositions 4 and 5, see Pages 243 and 244.]



## PROPOSITION 6. THEOREM.

*If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to the homologous sides.*

Let the triangles  $ABC$ ,  $DEF$  have the angle  $BAC$  equal to the angle  $EDF$ , and the sides about those angles proportionals, namely,  $BA$  to  $AC$  as  $ED$  is to  $DF$ :  
*the  $\triangle ABC$  shall be equiangular to the  $\triangle DEF$ , and shall have the  $\angle ABC$  equal to the  $\angle DEF$ , and the  $\angle ACB$  equal to the  $\angle DFE$ .*



**Construction.** At  $D$  make the  $\angle FDG$  equal to either of the angles  $BAC$ ,  $EDF$ , the points  $G$  and  $E$  being on opposite sides of  $DF$ ; and at  $F$  make the  $\angle DFG$  equal to the  $\angle ACB$ .  
 [I. 23.]

**Proof.** The remaining  $\angle$  at  $G$  = the remaining  $\angle$  at  $B$ .  
 $\therefore$  the  $\triangle DGF$  is equiangular to the  $\triangle ABC$ ;

$$\therefore GD : DF :: BA : AC, \quad [\text{VI. 4.}]$$

$$\text{that is, } :: ED : DF; \quad [\text{Hyp. and V. 11.}]$$

$$\therefore ED = GD \quad [\text{V. 9.}]$$

Then, in the triangles  $EDF$ ,  $GDF$ ,

because  $\begin{cases} ED = GD, \\ \text{and } DF \text{ is common,} \\ \text{and the included } \angle EDF = \text{the included } \angle GDF; \end{cases}$

$\therefore$  the base  $EF$  = the base  $GF$ , the  $\angle DFG$  = the  $\angle DFE$ ,  
 and the  $\angle$  at  $G$  = the  $\angle$  at  $E$ .  
 [I. 4.]

But the  $\angle DFG$  = the  $\angle ACB$ ; [Construction.]

$\therefore$  the  $\angle ACB$  = the  $\angle DFE$ . [Axiom 1.]

Also the  $\angle BAC = \angle EDF$ ;

[*Hypothesis.*

$\therefore$  the remaining  $\angle$  at  $B =$  the remaining  $\angle$  at  $E$ ;

$\therefore$  the  $\triangle ABC$  is equiangular to the  $\triangle DEF$ .

Wherefore, *if two triangles, etc.*

[Q. E. D.]

### EXERCISES ON PROPOSITION 4.

**\*\*1.** Every straight line drawn parallel to the base of a triangle cuts off a similar triangle.

**2.** If from a point  $A$  are drawn two straight lines, one to touch a circle in  $B$ , and the other to cut it in  $C$  and  $D$ , the triangles  $ABC$ ,  $ADB$  are similar.

**3.** The side  $BC$  of a triangle  $ABC$  is produced to  $D$  so that the triangles  $ABD$ ,  $ACD$  are similar. Prove that  $AD$  touches the circum-circle of the triangle  $ABC$ .

**4.** Two chords  $AB$ ,  $CD$  of a circle are produced towards  $B$  and  $D$  to meet at  $E$ ;  $CB$  is produced to meet at  $F$  a line through  $E$  parallel to  $AD$ ; prove that  $BF$ ,  $EF$  and  $CF$  are proportional.

**5.** Two circles intersect in  $A$  and the tangents at  $A$  to them meet the circles again in  $B$  and  $C$ ; prove that  $AB$  is to  $AC$  in the ratio of the two radii.

**6.** Two chords  $AB$ ,  $CD$  of a circle intersect at a point  $E$ , either within or without a circle; prove that  $AEC$ ,  $BED$  are similar triangles and also  $AED$ ,  $BEC$ .

**7.**  $ABC$  is a triangle having the angle  $C$  double of the angle  $B$ ; if the bisector of  $C$  meet  $AB$  in  $D$ , prove that  $AC$  is a mean proportional between  $AD$  and  $AB$ .

[The two triangles  $ACD$ ,  $ABC$  are similar.]

**8.**  $ABCD$  is a parallelogram; through  $D$  a straight line is drawn to cut  $BA$  and  $BC$  produced in  $E$  and  $F$ . Show that  $EAD$  and  $DCF$  are similar triangles.

**9.**  $D$  is a point in the base  $BC$  of a triangle  $ABC$  and  $EFG$  is a straight line parallel to  $BC$  which cuts  $AB$ ,  $AD$ ,  $AC$  in  $E$ ,  $F$ ,  $G$  respectively; prove that  $EF$  is to  $FG$  as  $BD$  to  $DC$ .

**\*\*10.** The diameters of the circles circumscribing the triangles formed by joining the vertex with any point in the base of a given triangle are proportional to the sides of that triangle.

**11.**  $ABCD$  is a parallelogram, and through  $D$  is drawn a straight line to meet  $AB$  in  $E$  and  $BC$  in  $F$ ; prove that  $EA$ ,  $AB$ ,  $AD$ , and  $CF$  are proportionals.

## EXERCISES ON PROPOSITIONS 5, 6.

1. AB and CD are two parallel straight lines; E is the middle point of CD; AC and BE meet at F, and AE and BD meet at G: shew that FG is parallel to AB.

2. A, B, C are three fixed points in a straight line; any straight line is drawn through C; shew that the perpendiculars on it from A and B are in a constant ratio.

3. If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

[This fixed point divides the line joining the two given points in the given ratio.]

4. Find a straight line such that the perpendiculars on it from three given points shall be in given ratios to each other.

[Use Ex. 3, and assume the construction of Prop. 10.]

5. C is the centre of a circle, and A any point within it; CA is produced through A to a point B such that the radius is a mean proportional between CA and CB: shew that if P be any point on the circumference, the angles CPA and CBP are equal.

6. In the figure of I. 43 shew that if EG and FH be produced they will meet on AC produced.

7. Two fixed circles touch at A and P and P' are points, one on each circle, such that PAP' is a right angle; prove that PP' always meets the line joining the centres of the circles in a fixed point.

[Let PP' meet the straight line joining the centres O and O' in Q; then  $\angle QOP' = \text{twice } \angle QAP' = \text{twice complement of } \angle PAO = \angle AOP$ ;  $\therefore \triangle^s QOP', QOP$  are similar;  $\therefore OQ : QO' :: OP : O'P'$ , etc.]

8. ABCD is a parallelogram and DEF a straight line cutting AB in E and CB produced in F; prove that the triangle AEF is a mean proportional to the triangles AED, BEF.

[The  $\triangle^s$  AED, FEB are similar;  $\therefore ED : EF :: AE : EB$ , etc.]

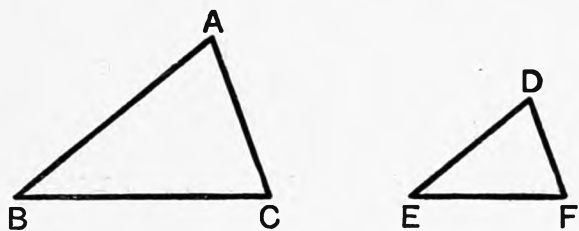
9. Through D, any point in the base of a triangle ABC, straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: shew that the triangle AEF is a mean proportional between the triangles FBD, EDC.

## PROPOSITION 7. THEOREM.

*If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle of each proportional, the sides opposite the equal angles being homologous, the third angles of the triangles shall be either equal or supplementary.*

Let the triangles  $ABC$ ,  $DEF$  have the angles  $ABC$ ,  $DEF$  equal, and the sides about the angles  $A$  and  $D$  proportional, so that  $BA$  is to  $AC$  as  $ED$  to  $DF$ :

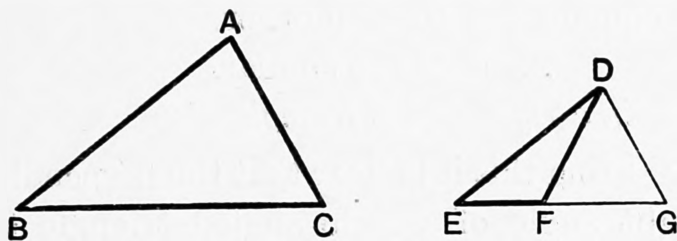
*the angles  $ACB$ ,  $DFE$  shall be equal or supplementary.*



*First*, let the angles  $BAC$ ,  $EDF$  be equal.

Then the third  $\angle ACB =$  the third  $\angle DFE$ . [I. 32.]

*Next*, let  $BAC$  be the greater of the two angles  $BAC$ ,  $EDF$ , At the point  $D$  make the  $\angle EDG$  equal to the  $\angle BAC$ , and produce  $EF$  to meet  $DG$  in  $G$ ;



$\therefore$  the third  $\angle DGE$  of the  $\triangle DGE =$  the third  $\angle ACB$ , and the triangles  $ABC$ ,  $DEG$  are equiangular;

$\therefore BA : AC :: ED : DG$ . [VI. 4.]

But  $BA : AC :: ED : DF$ ;

[Hypothesis.]

$\therefore DG = DF$ , and the  $\angle DGF =$  the  $\angle DFG$ .

But the  $\angle DGF =$  the  $\angle ACB$ , since  $ABC$ ,  $DEG$  are equiangular; and  $DFG$ ,  $DFE$  are supplementary angles;

$\therefore ACB$  and  $DFE$  are supplementary angles.

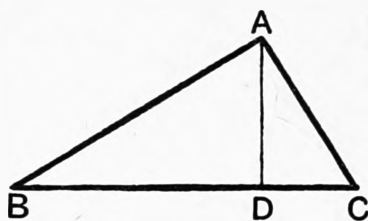
Wherefore, *if two triangles, etc.*

[Q. E. D.]

## PROPOSITION 8. THEOREM.

*In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.*

Let  $ABC$  be a triangle, having a right angle  $BAC$ , and let  $AD$  be drawn perpendicular to the base  $BC$ :  
*the triangles  $DBA$ ,  $DAC$  shall be similar to the triangle  $ABC$ , and to one another.*



**Proof.** The right  $\angle BAC =$  the right  $\angle BDA$ , [Axiom 11.  
 and the  $\angle$  at  $B$  is common to the two triangles  $ABC$ ,  $DBA$ ;  
 $\therefore$  the remaining  $\angle ACB =$  the remaining  $\angle DAB$ .  
 $\therefore$  the triangles  $ABC$ ,  $DBA$  are equiangular;  
 $\therefore$  they are similar. [VI. 4, Corollary.]

Similarly, the  $\triangle DAC$  is similar to the  $\triangle ABC$ .  
 $\therefore$  the triangles  $DBA$ ,  $DAC$  being both equiangular to the  $\triangle ABC$ , are equiangular to each other.  
 $\therefore$  they are similar.

Wherefore, *in a right-angled triangle, etc.* [Q. E. D.]

**Corollary.** From this it is clear that the perpendicular drawn from the right angle of a right-angled triangle to the base is a mean proportional between the segments of the base, and that each of the sides is a mean proportional between the base and the segment of the base adjacent to that side.

For, in the triangles  $DBA$ ,  $DAC$ ,

$BD : DA :: DA : DC$ ; [VI. 4.]

and in the triangles  $ABC$ ,  $DBA$ ,

$BC : BA :: BA : BD$ ; [VI. 4.]

and in the triangles  $ABC$ ,  $DAC$ ,

$BC : CA :: CA : CD$ . [VI. 4.]

**EXERCISES.**

**\*\*1.** The three sides of a right-angled triangle and the perpendicular from the right angle on the hypotenuse are proportionals

**2.** If one side of a right-angled triangle be double the other, prove that the perpendicular from the vertex on the hypotenuse divides it in the ratio 1 to 4.

**3.** ABC is a triangle, and a perpendicular is drawn from A to the opposite side, meeting it at D between B and C: shew that if AD is a mean proportional between BD and CD, the angle BAC is a right angle.

**4.** ABC is a triangle, and a perpendicular is drawn from A on the opposite side, meeting it at D between B and C: shew that if BA is a mean proportional between BD and BC, the angle BAC is a right angle.

**5.** Two straight lines are drawn from a point A to touch a circle of which the centre is E; the points of contact are joined by a straight line which cuts EA at H; and on HA as diameter a circle is described: shew that the straight lines drawn through E to touch this circle will meet it on the circumference of the given circle.

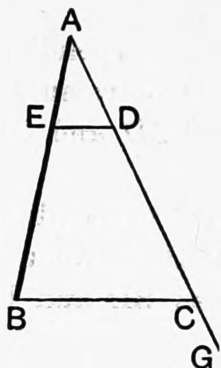
**6.** ABC is a triangle, right-angled at A, and BP is drawn perpendicular to the tangent at A to the circumcircle of ABC; prove that BP, BA, and BC are proportionals.

## PROPOSITION 9. PROBLEM.

*From a given straight line to cut off any assigned sub-multiple.*

Let  $AB$  be the given straight line :

*it is required to cut off any assigned sub-multiple.*



**Construction.** From the point  $A$  draw a straight line  $AG$ , making any angle with  $AB$  ;  
in  $AG$  take any point  $D$ , and take  $AC$ , the same multiple of  $AD$ , that  $AB$  is of the part which is to be cut off from it ;  
join  $BC$ , and draw  $DE$  parallel to it.  $AE$  shall be the part required to be cut off.

**Proof.** Because  $ED$  is parallel to the side  $BC$  of the  $\triangle ABC$  ;

[Construction.

$\therefore BE : EA :: CD : DA$ ,

[VI. 2.

and, by composition,  $BA : AE :: CA : AD$  ;

[V. 18.

$\therefore BA$  is the same multiple of  $AE$  that  $CA$  is of  $AD$ , [V. D.  
that is,  $AE$  is the required sub-multiple.

Wherefore, *from the given straight line  $AB$  the assigned sub-multiple has been cut off.*

[Q.E.F.

## EXERCISES.

Trisect a given straight line.

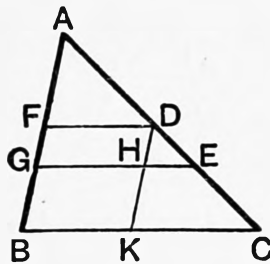
Cut off one-fifth of a given straight line.

Cut off four-sevenths of a given straight line.

## PROPOSITION 10. PROBLEM.

*To divide a given straight line similarly to a given divided straight line.*

Let  $AB$  be the straight line given to be divided, and  $AC$  the given divided straight line :  
*it is required to divide  $AB$  similarly to  $AC$ .*



**Construction.** Let  $AC$  be divided at the points  $D, E$ ; and let  $AB, AC$  be placed so as to contain any angle, and join  $BC$ ; through  $D$  draw  $DF$  parallel to  $BC$ , and through  $E$  draw  $EG$  parallel to  $BC$ . [I. 31.]

$AB$  shall be divided at  $F$  and  $G$  similarly to  $AC$ .

Through  $D$  draw  $DHK$  parallel to  $AB$ . [I. 31.]

**Proof.** Each of the figures  $FH, HB$  is a parallelogram ;  
 $\therefore DH = FG$ , and  $HK = GB$ . [I. 34.]

Then, because  $HE$  is parallel to  $KC$ , [Construction.]

$\therefore KH : HD :: CE : ED$ . [VI. 2.]

But  $KH = BG$ , and  $HD = GF$ ;

$\therefore BG : GF :: CE : ED$ . [V. 7.]

Again, because  $FD$  is parallel to  $GE$ , [Construction.]

$\therefore GF : FA :: ED : DA$ . [VI. 2.]

And it has been shewn that  $BG : GF :: CE : ED$ .

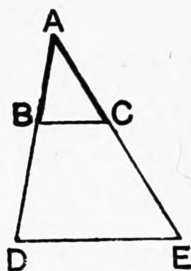
Wherefore the given straight line  $AB$  is divided similarly to the given divided straight line  $AC$ . [Q.E.F.]



## PROPOSITION 11. PROBLEM.

*To find a third proportional to two given straight lines.*

Let  $AB$ ,  $AC$  be the two given straight lines :  
*it is required to find a third proportional to  $AB$ ,  $AC$ .*



**Construction.** Let  $AB$ ,  $AC$  be placed so as to contain any angle ; produce  $AB$ ,  $AC$  to  $D$ ,  $E$  ; and make  $BD$  equal to  $AC$  ; [I. 3.]

join  $BC$ , and through  $D$  draw  $DE$  parallel to  $BC$ . [I. 31.]

$CE$  shall be a third proportional to  $AB$ ,  $AC$ .

**Proof.** Because  $BC$  is parallel to  $DE$ , a side of the  $\triangle ADE$  ; [Construction.]

$$\therefore AB : BD :: AC : CE ; \quad [\text{VI. 2.}]$$

$$\text{but } BD = AC ; \quad [\text{Construction.}]$$

$$\therefore AB : AC :: AC : CE.$$

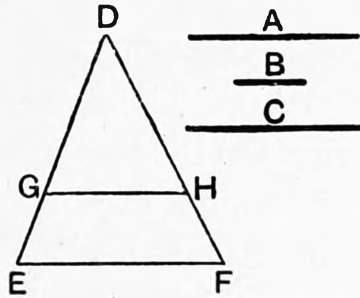
Wherefore, to the given straight lines  $AB$ ,  $AC$ , a third proportional  $CE$  is found. [Q. E. F.]

## PROPOSITION 12. PROBLEM.

*To find a fourth proportional to three given straight lines.*

Let  $A$ ,  $B$ ,  $C$  be the three given straight lines :

*it is required to find a fourth proportional to  $A$ ,  $B$ ,  $C$ .*



**Construction.** Take two straight lines,  $DE$ ,  $DF$ , containing any angle  $EDF$  ; and in these make  $DG$  equal to  $A$ ,  $GE$  equal to  $B$ , and  $DH$  equal to  $C$  ; [I. 3.

join  $GH$ , and through  $E$  draw  $EF$  parallel to  $GH$  to meet  $DF$  in  $F$ . [I. 31.

$HF$  shall be a fourth proportional to  $A$ ,  $B$ ,  $C$ .

**Proof.** Because  $GH$  is parallel to  $EF$ , a side of the  $\triangle DEF$ , [Construction.

$\therefore DG : GE :: DH : HF$ . [VI. 2.

But  $DG = A$ ,  $GE = B$ , and  $DH = C$  ; [Construction.

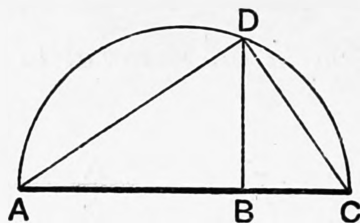
$\therefore A : B :: C : HF$ .

Wherefore, to the three given straight lines  $A$ ,  $B$ ,  $C$ , a fourth proportional  $HF$  is found. [Q. E. F.

## PROPOSITION 13. PROBLEM.

*To find a mean proportional between two given straight lines.*

Let  $AB$ ,  $BC$  be the two given straight lines:  
*it is required to find a mean proportional between them.*



**Construction.** Place  $AB$ ,  $BC$  in a straight line, and on  $AC$  describe the semicircle  $ADC$ ;

from  $B$  draw  $BD$  at right angles to  $AC$ . [I. 11.]

$BD$  shall be a mean proportional between  $AB$  and  $BC$ .

Join  $AD$ ,  $DC$ .

**Proof.** The  $\angle ADC$ , being in a semicircle, is a right  $\angle$ , [III. 31. and  $DB$  is drawn from the right  $\angle$  perpendicular to the base;  $\therefore DB$  is a mean proportional between  $AB$ ,  $BC$ , the segments of the base. [VI. 8, Corollary.]

Wherefore, *between the two given straight lines  $AB$ ,  $BC$ , a mean proportional  $DB$  is found.* [Q.E.F.]

## EXERCISES.

**\*\*1.** If two circles touch each other, and also touch a given straight line, the part of the straight line between the points of contact is a mean proportional between the diameters of the circles.

[Let the circles, centres  $O$  and  $O'$ , touch one another in  $A$  and the straight line in  $P$ ,  $P'$ ; draw the tangent at  $A$  to meet  $PP'$  in  $T$ . Then  $TP = TA = TP'$ . Also,  $OTO'$  is a right  $\angle$ , since  $OT$ ,  $O'T$  bisect  $\angle$ s  $PTA$ ,  $P'TA$ .  $\therefore TA$  is a mean proportional between  $AO$ ,  $AO'$ , etc.]

**2.** The bisector of the angle  $A$  of a triangle meets the base  $BC$  in  $D$  and the straight line bisecting  $BC$  at right angles in  $E$ ; prove that  $EB$  is a mean proportional between  $EA$  and  $ED$ .

**3.**  $AB$ ,  $AC$  are equal chords of a circle, and  $AED$  is any chord meeting  $BC$  in  $E$ ; prove that  $AB$  is a mean proportional between  $AD$  and  $AE$ .

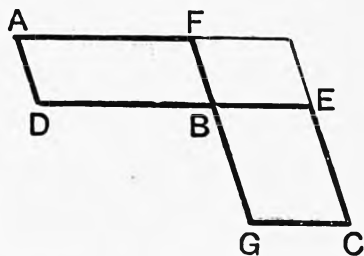
**4.** Two circles intersect in  $A$  and  $B$ , and the tangents at  $A$  to the two circles meet the circumferences in  $C$  and  $D$ ; prove that  $AB$  is a mean proportional between  $DB$  and  $BC$ .

## PROPOSITION 14. THEOREM.

*Parallelograms of equal area, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional ;*

*Conversely, parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal in area.*

Let AB, BC be parallelograms of equal area, which have the angle FBD equal to the angle EBG :  
*the sides of the parallelograms about the equal angles shall be reciprocally proportional, that is, DB shall be to BE as GB to BF.*



**Construction.** Let the parallelograms be placed, so that the sides DB, BE may be in the same straight line ;

$\therefore$  also FB, BG are in one straight line.

[I. 14.]

Complete the parallelogram FE.

**Proof.** Because the  $\parallel^{\text{ms}}$  AB and BC are equal, [Hypoth.  
 and that FE is another parallelogram ;

$\therefore$  AB : FE :: BC : FE.

But AB : FE :: the base DB : the base BE, [VI. 1.]

and BC : FE :: the base GB : the base BF ; [VI. 1.]

$\therefore$  DB : BE :: GB : BF. [V. 11.]

*Conversely*, let the angle FBD = the angle EBG, and let DB be to BE as GB to BF :

*the parallelograms AB and BC shall be equal in area.*

For, the same construction being made,

because DB : BE :: GB : BF, [Hypothesis.]

and that DB : BE :: the  $\parallel^{\text{m}}$  AB : the  $\parallel^{\text{m}}$  FE, [VI. 1.]

and that GB : BF :: the  $\parallel^{\text{m}}$  BC : the  $\parallel^{\text{m}}$  FE ; [VI. 1.]

$\therefore$  AB : FE :: BC : FE ; [V. 11.]

$\therefore$  the parallelogram AB = the parallelogram BC.

Wherefore, equal parallelograms, etc.

[Q. E. D.]

## PROPOSITION 15. THEOREM.

*Triangles of equal area, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.*

*Conversely, triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal in area.*

Let  $ABC$ ,  $ADE$  be triangles of equal area which have the angle  $BAC$  equal to the angle  $DAE$  :

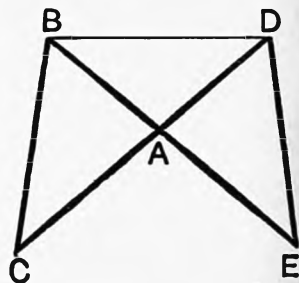
*the sides of the triangles about the equal angles shall be reciprocally proportional, that is,  $CA$  shall be to  $AD$  as  $EA$  to  $AB$ .*

**Construction.** Let the triangles be placed so that the sides  $CA$ ,  $AD$  may be in the same straight line ;

therefore also  $EA$ ,  $AB$  are in one straight line ;

[I. 14.]

join  $BD$ .



**Proof.** Because the  $\triangle ABC = \triangle ADE$ ,  
and that  $ABD$  is another  $\triangle$  ;

[Hypothesis.]

$\therefore \triangle ABC : \triangle ABD :: \triangle ADE : \triangle ABD$ .

[V. 7.]

But  $\triangle ABC : \triangle ABD :: \text{base } CA : \text{base } AD$ ,

[VI. 1.]

and  $\triangle ADE : \triangle ADB :: \text{base } EA : \text{base } AB$  ;

[VI. 1.]

$\therefore CA : AD :: EA : AB$ .

[V. 11.]

*Conversely*, let the  $\angle BAC = \angle DAE$ ,  
and let  $CA$  be to  $AD$  as  $EA$  to  $AB$  :  
*the triangles  $ABC$ ,  $ADE$  shall be equal in area.*

For, the same construction being made,  
because  $CA : AD :: EA : AB$ ,

[Hypothesis.]

and that  $CA : AD :: \triangle ABC : \triangle ABD$ ,

[VI. 1.]

and that  $EA : AB :: \triangle ADE : \triangle ABD$  ;

[VI. 1.]

$\therefore \triangle ABC : \triangle ABD :: \triangle ADE : \triangle ABD$  ;

[V. 11.]

$\therefore \triangle ABC = \triangle ADE$ .

[V. 9.]

Wherefore, *equal triangles, etc.*

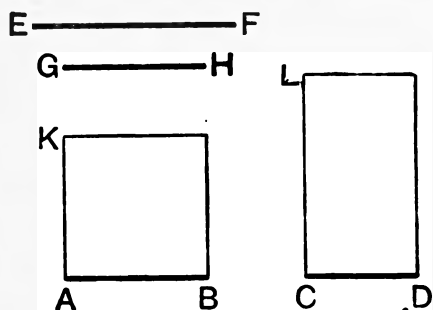
[Q. E. D.]

## PROPOSITION 16. THEOREM.

*If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.*

*Conversely, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.*

Let the four straight lines AB, CD, EF, GH be proportionals, namely, AB to CD as EF to GH:  
*the rectangle contained by AB and GH shall be equal to the rectangle contained by CD and EF.*



**Construction.** From the points A, C draw AK, CL at right angles to AB, CD; [I. 11.  
 make AK equal to GH and CL to EF, [I. 3.  
 and complete the parallelograms BK, DL. [I. 31.]

**Proof.** Because  $AB : CD :: EF : GH$ , [Hypothesis.  
 and that  $EF = CL$ , and  $GH = AK$ ; [Construction.]

$\therefore AB : CD :: CL : AK$ ,  
 that is, the sides of the parallelograms BK, DL about the equal angles are reciprocally proportional;

$\therefore BK = DL$ . [VI. 14.]

But  $BK = \text{the rect. } AB, GH$ , since  $AK = GH$ ,  
 and  $DL = \text{the rect. } CD, EF$ , since  $CL = EF$ ;  
 $\therefore \text{the rect. } AB, GH = \text{the rect. } CD, EF$ .

*Conversely, let the rectangle AB, GH = the rect. CD, EF:  
 then shall AB be to CD as EF to GH.*

For, the same construction being made,  
 because the rect. AB, GH = the rect. CD, EF, [Hypothesis.]

and that  $BK = \text{the rect. } AB, GH$ , since  $AK = GH$ , [Const.  
 and that  $DL = \text{the rect. } CD, EF$ , since  $CL = EF$ ; [Const.  
 $\therefore BK = DL$ . [Axiom 1.

And these parallelograms are equiangular;

$\therefore$  the sides about their equal angles are reciprocally proportional, that is,  $AB : CD :: CL : AK$ . [VI. 14.

But  $CL = EF$ , and  $AK = GH$ ;

$\therefore AB : CD :: EF : GH$ .

Wherefore, if four straight lines, etc.

[Q. E. D.]

### EXERCISES.

1. Shew that the diagonals of any quadrilateral figure inscribed in a circle divide the quadrilateral into four triangles which are similar two and two; and deduce the theorem of III. 35.

2. The exterior angle  $A$  of a triangle  $ABC$  is bisected by a straight line which meets the base in  $D$  and the circumcircle of the triangle in  $E$ ; prove that the rectangle  $BA, AC = \text{the rectangle } EA, AD$ .

3.  $ABC$  is a triangle. In  $AB, AC$  are taken points  $D, E$  so that  $AB$  is to  $AC$  as  $AE$  to  $AD$ . Prove that the diameter of the circumscribing circle of the triangle  $ABC$  which passes through  $A$  is perpendicular to  $DE$ .

[ $AE \cdot AC = AB \cdot AD$ .  $\therefore BDEC$  is a cyclic quad<sup>l</sup>.

$\therefore \angle AED = \angle ABC = \text{rt. } \angle - \angle EAO$ ,

where  $O$  is the circumcentre of  $\triangle ABC$ .]

4. A straight line meets two intersecting circles in  $P, Q, R, S$  and their common chord in  $O$ ; prove that  $OP, OQ, OR, OS$ , taken in a suitable order, are proportionals.

5. If two lines,  $AD$  and  $AE$ , be drawn through the vertex equally inclined to the bisector of the vertical angle of a triangle  $ABC$  to meet the base and the circumcircle of the triangle in  $D$  and  $E$ , the rectangle contained by them is equal to the rectangle contained by the sides.

[The  $\triangle^s ABD, AEC$  are similar.]

6.  $I$  is the incentre of the triangle  $ABC$  and  $I_1$  the centre of the escribed circle opposite the angle  $A$ ; prove that the rectangles  $AI, AI_1$  and  $AB, AC$  are equal.

[ $ACI_1, AIB$  are similar  $\triangle^s$ .]

\*\*7.  $ABCD$  is a straight line; find a point  $P$  in  $A$  such that  $PA, PB, PC, PD$  are proportionals.

[On  $AC, BD$  describe semi-circles; they meet in a point such that the foot of the perpendicular from it upon  $AD$  is the required point  $P$ .]

## PROPOSITION 17. THEOREM.

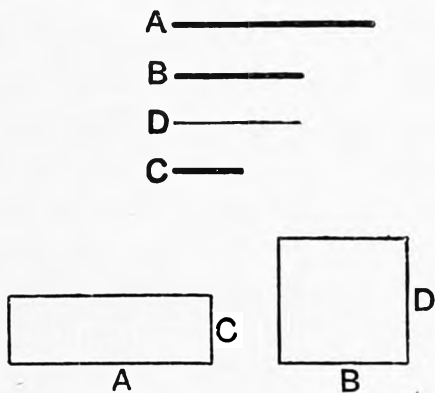
*If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean.*

*Conversely, if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportionals.*

Let the three straight lines A, B, C be proportionals, namely, let A be to B as B is to C :

*the rectangle contained by A and C shall be equal to the square on B.*

Take D equal to B.



**Proof.** Then, because  $A : B :: B : C$ ,

[Hypothesis.

and that  $B = D$ ;

$\therefore A : B :: D : C$ ;

[V. 7.

$\therefore$  the rect. A, C = the rect. B, D.

[VI. 16.

But the rect. B, D = the square on B, since  $D = B$ ;

[Const.

$\therefore$  the rect. A, C = the square on B.

*Conversely, let the rect. A, C = the square on B :*

*A shall be to B as B to C.*

For, the same construction being made,

because the rect. A, C = the square on B,

[Hypothesis.

and that the square on B = the rect. B, D, since  $D = B$ ;

[Const.

$\therefore$  the rect. A, C = the rect. B, D;

$\therefore A : B :: D : C$ ;

[VI. 16.

that is,  $A : B :: B : C$ .

Wherefore, *if three straight lines, etc.*

[Q.E.D.

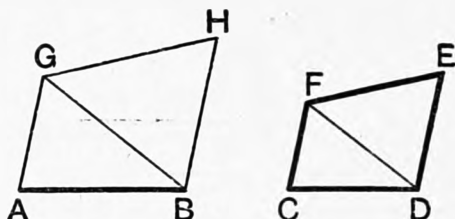


## PROPOSITION 18. PROBLEM.

*On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.*

Let  $AB$  be the given straight line, and  $CDEF$  the given rectilineal figure of four sides :

*it is required to describe on  $AB$  a rectilineal figure similar and similarly situated to  $CDEF$ .*



**Construction.** Join  $DF$  ; at the point  $A$  make the  $\angle BAG$  equal to the  $\angle DCF$  ; and at the point  $B$  make the  $\angle ABG$  equal to the  $\angle CDF$  ; [I. 23.]

$\therefore$  the remaining  $\angle AGB =$  the remaining  $\angle CFD$ ,  
and the  $\triangle AGB$  is equiangular to the  $\triangle CFD$ .

Again, at  $B$  make the  $\angle GBH$  equal to the  $\angle FDE$ ,

and at  $G$  make the  $\angle BGH$  equal to the  $\angle DFE$  ; [I. 23.]

$\therefore$  the remaining  $\angle BHG =$  the remaining  $\angle DEF$ ,  
and the  $\triangle BHG$  is equiangular to the  $\triangle DEF$ .

Then  $ABHG$  shall be the figure required.

**Proof.** Because the  $\angle AGB =$  the  $\angle CFD$ ,

and the  $\angle BGH =$  the  $\angle DFE$ , [Construction.]

$\therefore$  the whole  $\angle AGH =$  the whole  $\angle CFE$ .

For the same reason the  $\angle ABH =$  the  $\angle CDE$ .

Also the  $\angle BAG =$  the  $\angle DCF$ , and the  $\angle BHG =$  the  $\angle DEF$ .

$\therefore$  the figure  $ABHG$  is equiangular to the figure  $CDEF$ .

Also these figures have their sides about the equal angles proportionals.

For, because the  $\triangle BAG$  is equiangular to the  $\triangle DCF$ ,

$\therefore BA : AG :: DC : CF$ . [VI. 4.]

Also, for the same reason,  $AG : GB :: CF : FD$ , and

$$BG : GH :: DF : FE;$$

$\therefore$  *ex æquali*,  $AG : GH :: CF : FE$ . [V. 22.]

Similarly,  $AB : BH :: CD : DE$ ,

and  $GH : HB :: FE : ED$ . [VI. 4.]

$\therefore$  ABHG and CDEF are equiangular, and have their sides about the equal angles proportionals;

$\therefore$  they are similar. [VI. Definition 1.]

The construction and proof would be similar to the above whatever be the number of sides that the given figure has.

### EXERCISES ON PROPOSITION 17.

**\*\*1.** If the tangents to a circle at P and Q meet in T, and if C be the centre and CT meet PQ in N and the circle in A, prove that the rect. CN, CT = the square on CA.

**\*\*2.** If there be drawn two parallel tangents to a circle, and any variable tangent touch the circle at B and meet the parallel tangents in A and C, the rectangle AB, BC is equal to the square on the radius of the circle.

**3.** If AD be the tangent at a point D of a circle, and DE be the diameter through D, and AE be joined to cut the circle in F, prove that the rectangle AE, EF is the same for all positions of A.

**4.** The straight line bisecting at right angles AC, one of the equal sides of an isosceles triangle, meets the base BC produced in D. Prove that AC is a mean proportional between CB and CD.

**\*\*5.** The rectangle contained by the perpendiculars from any point P of a circle on any two tangents is equal to the square of the perpendicular from the same point upon their chord of contact.

[Let B, C be the points of contact; PK, PL, PM the  $\perp^{\text{rs}}$  from P upon the tangents at B, C, and on BC. Then PKBM and PLCM are cyclic quadrilaterals;

$$\therefore \angle KMP = \angle KBP = \angle PCM = \angle PLM,$$

$$\text{and } \angle PKM = \angle PBM = \angle PCL = \angle PML;$$

$\therefore$  the  $\triangle^s$  PKM, PLM are similar;  $\therefore$  etc.]

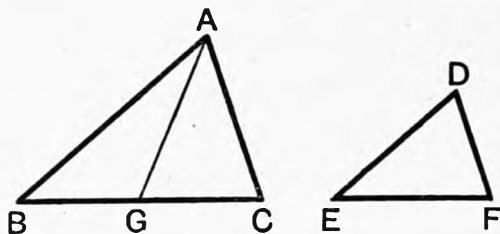
**\*\*6.** ABC is a straight line; find a point P in it such that PB may be a mean proportional between PA and PC.

[On AC describe a semi-circle; at B draw a str. line BQ such that  $\angle ABQ = \text{half a right } \angle$ ; let BQ meet the circle in Q; draw QP perpendicular to AB, etc.]

## PROPOSITION 19. THEOREM.

*Similar triangles are to one another in the duplicate ratio of their homologous sides.*

Let  $ABC$  and  $DEF$  be similar triangles, having the angle  $B$  equal to the angle  $E$ , and let  $AB$  be to  $BC$  as  $DE$  is to  $EF$ , so that the side  $BC$  is homologous to the side  $EF$ :  
*the  $\triangle ABC$  shall be to the  $\triangle DEF$  in the duplicate ratio of  $BC$  to  $EF$*



**Construction.** Take  $BG$  a third proportional to  $BC$  and  $EF$ , so that  $BC$  may be to  $EF$  as  $EF$  to  $BG$ , [VI. 11.]  
 and join  $AG$ .

**Proof.** Because  $AB : BC :: DE : EF$ ; [Hypothesis.  
 $\therefore$  alternately,  $AB : DE :: BC : EF$ ; [V. 16.  
 but  $BC : EF :: EF : BG$ ; [Construction.  
 $\therefore AB : DE :: EF : BG$ , [V. 11.]

that is, the sides of the triangles  $ABG$  and  $DEF$ , about their equal angles, are reciprocally proportional;

$\therefore$  the  $\triangle ABG$  is equal in area to the  $\triangle DEF$ . [VI. 15.]

Also, because  $BC : EF :: EF : BG$ ;

$\therefore BC$  is to  $BG$  in the duplicate ratio of  $BC$  to  $EF$ . [V. Def. 10.]

But the  $\triangle ABC : \text{the } \triangle ABG :: BC : BG$ ; [VI. 1.]

$\therefore$  the  $\triangle ABC$  is to the  $\triangle ABG$  in the duplicate ratio of  $BC$  to  $EF$ .

But the  $\triangle ABG$  was shewn equal to the  $\triangle DEF$ ;

$\therefore$  the  $\triangle ABC$  is to the  $\triangle DEF$  in the duplicate ratio of  $BC$  to  $EF$ .

Wherefore, *similar triangles, etc.*

[Q. E. D.]

**Corollary.** From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any  $\triangle$  described on the first to a similar and similarly described  $\triangle$  on the second.

## EXERCISES.

[In the following examples the student may assume the result of Cor. 3 to Prop. 20.]

1. Make a triangle four times as large as a given triangle and similar to it.

[If  $ABC$  be the given  $\triangle$ , produce  $AB$ ,  $AC$  to  $P$ ,  $Q$  so that  $AP = \text{twice } AB$  and  $AQ = \text{twice } AC$ . Then  $APQ$  is the required  $\triangle$  by VI. 19 and 20, Cor. 2.]

2. Bisect the area of a triangle by a straight line drawn parallel to one of the sides.

[Let the required line meet the sides  $AB$ ,  $AC$  in  $P$ ,  $Q$ ; then by VI. 19 and 20, Cor. 2,  $AP^2 : AB^2 :: 1 : 2$ .  $\therefore AB^2 = 2AP^2$ . To construct  $AP$ , on  $AB$  as diameter, describe a semicircle, and let  $R$  be its middle point; then  $AB^2 = AR^2 + RB^2 = 2AR^2$ .  $\therefore AP = AR$ . Hence the construction.]

3. Construct a triangle twice as large as a given triangle and similar to it. [Here  $AP^2 = 2AB^2$ . Use I. 47.]

4. Construct a triangle five times as large as a given triangle and similar to it. [Here  $AP^2 = 5AB^2$ .]

5.  $ABC$  is a triangle and  $BE$ ,  $CF$  are perpendiculars upon the sides  $AC$ ,  $AB$ . Prove that the triangle  $ABE$  is to the triangle  $ACF$  as the square on  $AB$  to the square on  $AC$ .

6.  $AB$  and  $CD$  are two chords of a circle intersecting in an external point  $E$ ; prove that the triangle  $EAC$  is to the triangle  $EBD$  as  $AC^2$  to  $BD^2$ .

7.  $O$  is a point without a circle, centre  $C$ ,  $OT$  is a tangent and  $OPCQ$  a straight line cutting the circle;  $PN$  is another tangent cutting  $OT$  in  $N$ ; shew that the triangle  $OPN$ ,  $OTC$  are to one another in the ratio  $OP$  to  $OQ$ .

8.  $ABC$  is a triangle, and  $AD$ ,  $BE$ ,  $CF$  the perpendiculars on the opposite sides. Prove that the areas of the  $\triangle^s DBF$ ,  $ABC$  are to one another as the squares on  $BD$ ,  $AB$ .

9. Prove that the area of the regular hexagon inscribed in a circle is three-quarters of the regular hexagon described about the same circle.

10. Prove that similar triangles are to one another in the duplicate ratio of corresponding medians, and also in the duplicate ratio of the radii of their circumcircles.

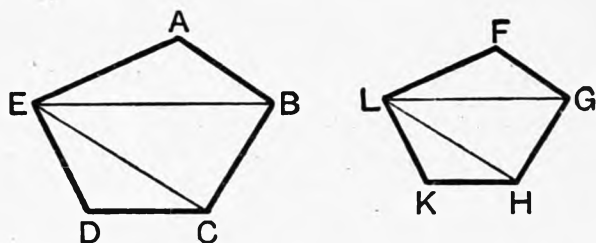
## PROPOSITION 20. THEOREM.

*Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have ; and the polygons are to one another in the duplicate ratio of their homologous sides.*

Let  $ABCDE$ ,  $FGHKL$  be similar polygons, and let  $AB$  be the side homologous to the side  $FG$  :

(1) *the polygons  $ABCDE$ ,  $FGHKL$  may be divided into the same number of similar triangles, of which each shall have to each the same ratio which the polygons have ; and*

(2) *the polygon  $ABCDE$  shall be to the polygon  $FGHKL$  in the duplicate ratio of  $AB$  to  $FG$ .*



**Construction.** Join  $BE$ ,  $EC$ ,  $GL$ ,  $LH$ .

**Proof.** (1) Because the polygon  $ABCDE$  is similar to the polygon  $FGHKL$ , [Hypothesis.

the  $\angle BAE =$  the  $\angle GFL$ , and  $BA : AE :: GF : FL$ ;

[VI. Definition 2.

$\therefore$  the  $\triangle ABE$  is equiangular to the  $\triangle FGL$ , [VI. 6.

and therefore these triangles are similar ; [VI. 4.

$\therefore$  the  $\angle ABE =$  the  $\angle FGL$ .

But, because the polygons are similar, [Hypothesis.

$\therefore$  the whole  $\angle ABC =$  the whole  $\angle FGH$  ; [VI. Definition 2.

$\therefore$  the remaining  $\angle EBC =$  the remaining  $\angle LGH$ . [Axiom 3.

Also, because the triangles  $ABE$  and  $FGL$  are similar,

$\therefore EB : BA :: LG : GF$  ;

and also, because the polygons are similar, [Hypothesis.

$\therefore AB : BC :: FG : GH$  ;

[VI. Definition 2.

$\therefore$  *ex æquali*,  $EB : BC :: LG : GH$ ,

[V. 22.

that is, the sides about the equal angles EBC and LGH are proportionals ;

$\therefore$  the  $\triangle EBC$  is equiangular to the  $\triangle LGH$  ; [VI. 6.]

$\therefore$  these triangles are similar. [VI. 4.]

For the same reason the  $\triangle ECD$  is similar to the  $\triangle LHK$ .

Therefore the similar polygons ABCDE, FGHLK may be divided into the same number of similar triangles.

(2) Again, because the  $\triangle ABE$  is similar to the  $\triangle FGL$ ,

$\therefore$  ABE is to FGL in the duplicate ratio of EB to LG. [VI. 19.]

For the same reason the  $\triangle EBC$  is to the  $\triangle LGH$  in the duplicate ratio of EB to LG ;

$\therefore \triangle ABE : \triangle FGL :: \triangle EBC : \triangle LGH$ . [V. 11.]

Again, because the  $\triangle EBC$  is similar to the  $\triangle LGH$ ,

$\therefore$  EBC is to LGH in the duplicate ratio of EC to LH. [VI. 19.]

For the same reason the  $\triangle ECD$  is to the  $\triangle LHK$  in the duplicate ratio of EC to LH ;

$\therefore \triangle EBC : \triangle LGH :: \triangle ECD : \triangle LHK$ . [V. 11.]

But it has been shewn that the  $\triangle EBC$  : the  $\triangle LGH$

$::$  the  $\triangle ABE$  : the  $\triangle FGL$  ;

$\therefore \triangle ABE : \triangle FGL :: \triangle EBC : \triangle LGH$ ,

and  $:: \triangle ECD : \triangle LHK$  ; [V. 11.]

and therefore as one of the antecedents is to its consequent, so are all the antecedents to all the consequents ; [V. 12.]

that is, as  $\triangle ABE : \triangle FGL$ ,

$::$  the polygon ABCDE : the polygon FGHLK.

But the  $\triangle ABE$  is to the  $\triangle FGL$  in the duplicate ratio of AB to FG ; [VI. 19.]

$\therefore$  the polygon ABCDE is to the polygon FGHLK in the duplicate ratio of AB to FG.

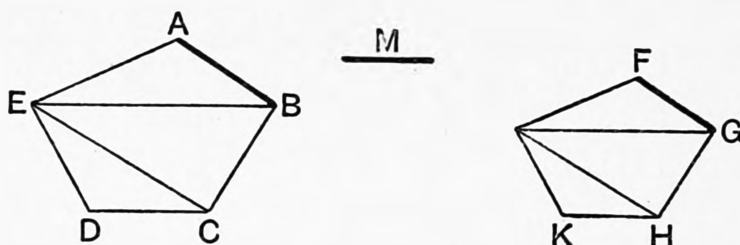
Wherefore, *similar polygons, etc.*

[Q.E.D.]

**Corollary 1.** In like manner it may be shewn that similar four-sided figures, or figures of any number of sides, are to one another in the duplicate ratio of their homologous sides ; and it has already been shewn for triangles ; therefore, universally,

*Similar rectilinear figures are to one another in the duplicate ratio of their homologous sides.*

**Corollary 2.** If to AB and FG, two of the homologous sides, a third proportional M be taken, [VI. 11.]



then AB has to M the duplicate ratio of AB to FG. [V. Def. 10.] But any rectilinear figure described on AB is to the similar and similarly described rectilinear figure on FG in the duplicate ratio of AB to FG. [Corollary 1.]

Therefore as  $AB : M ::$  the figure on AB : the figure on FG ; [V. 11.]

and this was shewn before for triangles. [VI. 19, Corollary.]

Wherefore, universally, *If three straight lines be proportionals, as the first is to the third, so is any rectilinear figure described on the first to a similar and similarly described rectilinear figure on the second.*

**Corollary 3.** If these similarly described figures be squares, it follows that the first : the third :: the square on the first : the square on the second.

But the first is to the third in the duplicate ratio of the first to the second. [V. Definition 10.]

Hence, *The duplicate ratio of two straight lines is equal to the ratio of the squares on the two straight lines.*

Propositions 19, 20 may therefore be enunciated thus :

*Similar triangles, and polygons, are to one another as the squares on their homologous sides.*

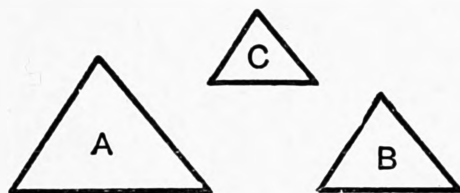


## PROPOSITION 21. THEOREM.

*Rectilineal figures which are similar to the same rectilineal figure are also similar to each other.*

Let each of the rectilineal figures A and B be similar to the rectilineal figure C :

*the figure A shall be similar to the figure B.*



**Proof.** Because A is similar to C,

[*Hypothesis.*

A is equiangular to C, and they have their sides about the equal angles proportionals.

[VI. *Definition 2.*

Again, because B is similar to C,

[*Hypothesis.*

B is equiangular to C, and they have their sides about the equal angles proportionals ;

[VI. *Definition 2.*

$\therefore$  the figures A and B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals ;

$\therefore$  A is equiangular to B,

[*Axiom 1.*

and they have their sides about the equal angles proportionals.

[V. 11.

$\therefore$  the figure A is similar to the figure B.

[VI. *Definition 2.*

Wherefore, *rectilineal figures, etc.*

[Q. E. D.



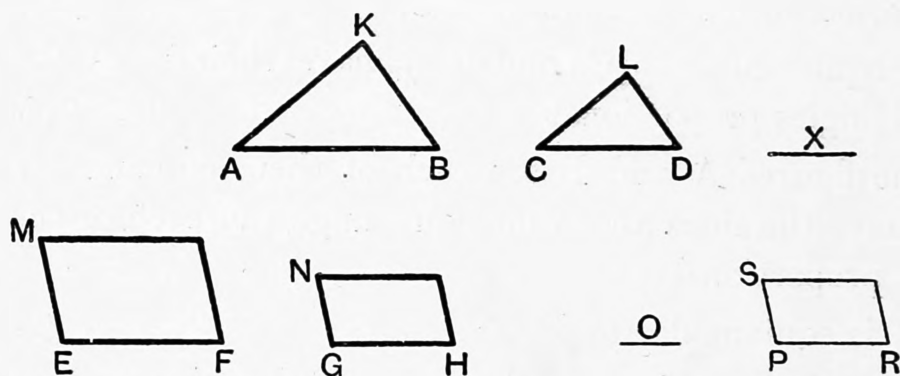
## PROPOSITION 22. THEOREM.

*If four straight lines be proportionals, and a pair of similar rectilineal figures be similarly described on the first and second, and also a pair on the third and fourth, these figures shall be proportionals.*

*Conversely, if a rectilineal figure on the first of four straight lines be to a similar and similarly described figure on the second as a rectilineal figure on the third is to a similar and similarly described figure on the fourth, these four straight lines shall be proportionals.*

Let the four straight lines AB, CD, EF, GH be proportionals, namely, AB to CD as EF is to GH; and on AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and on EF, GH let the similar rectilineal figures MF, NH be similarly described:

*the figure KAB shall be to the figure LCD as the figure MF is to the figure NH.*



**Construction.** To AB and CD take a third proportional X, and to EF and GH take a third proportional O.

**Proof.** Because  $AB : CD :: EF : GH$ ,

[Hypothesis.]

and  $AB : CD :: CD : X$ ,

[Construction.]

and  $EF : GH :: GH : O$ ;

[Construction.]

$\therefore CD : X :: GH : O$ .

[V. 11.]

Also  $AB : CD :: EF : GH$  ;

$\therefore$  *ex æquali*,  $AB : X :: EF : O$ . [V. 22.]

But  $AB : X :: KAB : LCD$  ;

and  $EF : O :: MF : NH$  ; [VI. 20, *Cor.* 2.]

$\therefore KAB : LCD :: MF : NH$ . [V. 11.]

*Conversely*, let  $KAB$  be to the similar figure  $LCD$  as  $MF$  to the similar figure  $NH$  :

$AB$  shall be to  $CD$  as  $EF$  to  $GH$ .

**Construction.** Make  $EF$  to  $PR$  as  $AB$  to  $CD$  ; [VI. 12.]  
and on  $PR$  describe  $SR$ , similar and similarly situated to either of the figures  $MF$ ,  $NH$ . [VI. 18.]

**Proof.** Because  $AB$  is to  $CD$  as  $EF$  to  $PR$ , and that on  $AB$ ,  $CD$  are described the similar figures  $KAB$ ,  $LCD$ , and on  $EF$ ,  $PR$  the similar figures  $MF$ ,  $SR$  ; therefore, by the former part of this proposition,

$$KAB : LCD :: MF : SR.$$

But, by hypothesis,  $KAB : LCD :: MF : NH$  ;

$$\therefore MF : SR :: MF : NH ; \quad [V. 11.]$$

$$\therefore SR = NH.$$

But  $SR$  and  $NH$  are similar and similarly situated ; [*Constr.*]

$$\therefore PR = GH.$$

And because  $AB : CD :: EF : PR$ , and that  $PR = GH$  ;

$$\therefore AB : CD :: EF : GH.$$

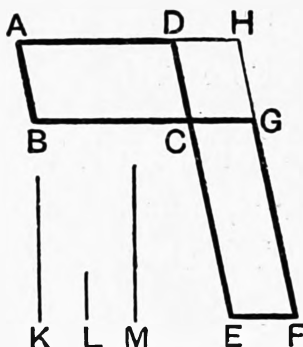
Wherefore, *if four straight lines, etc.*

[Q. E. D.]

## PROPOSITION 23. THEOREM.

*Parallelograms which are equiangular to one another have to one another the ratio which is compounded of the ratios of their sides.*

Let the parallelogram AC be equiangular to the parallelogram CF, having the  $\angle BCD$  equal to the  $\angle ECG$  :  
AC shall have to CF the ratio which is compounded of the ratios of their sides.



**Construction.** Let BC and CG be placed in a straight line ;  
 $\therefore$  DC and CE are also in a straight line ; [I. 14.  
complete the parallelogram DG ;  
take any straight line K, and make K to L as BC to CG,  
and L to M as DC to CE. [VI. 12.

**Proof.** The ratio of K to M is that which is compounded  
of the ratios of K to L and of L to M ; [V. Def. 12.

that is, of BC to CG and DC to CE.

Now, the  $\parallel^m$  AC : the  $\parallel^m$  CH :: BC : CG, [VI. 1.  
that is, :: K : L. [Constr.

Again, the  $\parallel^m$  CH : the  $\parallel^m$  CF :: DC : CE, [VI. 1.  
that is, :: L : M ; [Constr.

$\therefore$  *ex æquali*, the  $\parallel^m$  AC : the  $\parallel^m$  CF :: K : M. [V. 22.

But K has to M the ratio compounded of the ratios of the  
sides ;

$\therefore$  also the parallelogram AC has to the parallelogram CF  
the ratio compounded of the ratios of the sides.

Wherefore, *parallelograms, etc.*

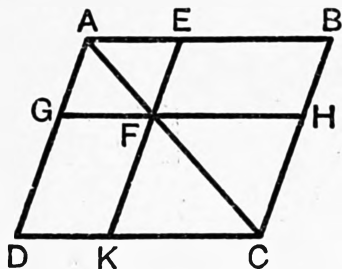
[Q. E. D.]

[An important note to this proposition will be found on Page 331.]

## PROPOSITION 24. THEOREM.

*Parallelograms about the diameter of any parallelogram are similar to the whole parallelogram, and to one another.*

Let ABCD be a parallelogram, of which AC is a diameter ; and let EG and HK be parallelograms about the diameter : EG and HK shall be similar both to the whole parallelogram and to one another.



**Proof.** Because DC and GF are parallel,  
the  $\angle ADC =$  the  $\angle AGF$ .

[I. 29.]

Because BC and EF are parallel,

$\therefore$  the  $\angle ABC =$  the  $\angle AEF$ .

Also each of the angles BCD and EFG is equal to the opposite angle BAD,

[I. 34.]

and therefore they are equal ;

$\therefore$  the parallelograms ABCD and AEFG are equiangular.

Again, because the  $\angle ABC =$  the  $\angle AEF$ , and the  $\angle BAC$  is common to the two triangles BAC and EAF ;

$\therefore$  these triangles are equiangular ;

$\therefore AB : BC :: AE : EF$ .

[VI. 4.]

Also the opposite sides of parallelograms are equal ;

[I. 34.]

$\therefore AB : AD :: AE : AG$ ,

and  $DC : CB :: GF : FE$ ,

and  $CD : DA :: FG : GA$ .

[V. 7.]

$\therefore$  the sides of ABCD and AEFG about their equal angles are proportional, and they are therefore similar. [VI. Def. 1.]

So also the parallelograms ABCD and FHCK are similar ;

$\therefore$  each of the parallelograms EG and HK is similar to BD ;

EG is similar to HK.

[VI. 21.]

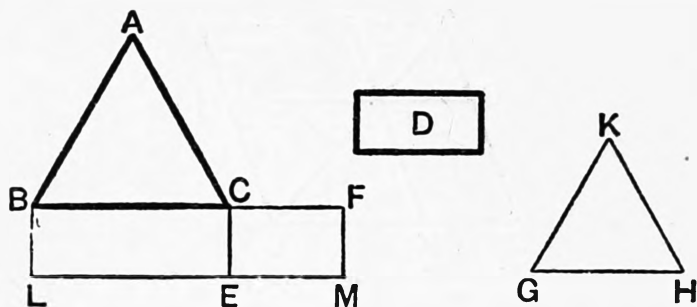
Wherefore, *parallelograms, etc.*

[Q.E.D.]

## PROPOSITION 25. PROBLEM.

*To describe a rectilineal figure which shall be similar to one given rectilineal figure and equal to another given rectilineal figure.*

Let  $ABC$  and  $D$  be the two given rectilineal figures :  
it is required to describe a rectilineal figure similar to  $ABC$  and equal to  $D$ .



**Construction.** On  $BC$  describe the parallelogram  $BE$  equal to the figure  $ABC$ .

On  $CE$  describe the parallelogram  $CM$  equal to  $D$ , and having the  $\angle FCE$  equal to the  $\angle CBL$ ; [I. 45, *Corollary*.]  
therefore  $BC$  and  $CF$  will be in one straight line, and  $LE$  and  $EM$  will be in one straight line.

Between  $BC$  and  $CF$  find a mean proportional  $GH$ , [VI. 13. and on  $GH$  describe the figure  $KGH$ , similar and similarly situated to the figure  $ABC$ . [VI. 18.]

$KGH$  shall be the figure required.

**Proof.** Because  $BC : GH :: GH : CF$ , [Construction.]

$\therefore BC : CF :: ABC : KGH$ . [VI. 20, *Corollary 2*.]

But  $BC : CF ::$  the  $\parallel^m BE : \text{the } \parallel^m CM$ ; [VI. 1.]

$\therefore ABC : KGH :: BE : CM$ . [V. 11.]

Also, the figure  $ABC = \text{the } \parallel^m BE$ ; [Construction.]

$\therefore$  the figure  $KGH = \text{the } \parallel^m CM$ , [V. 14.]

that is,  $= D$ , [Construction.]

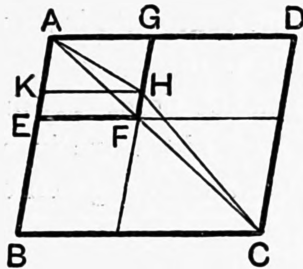
and it is similar to the figure  $ABC$ . [Construction.]

Wherefore the rectilineal figure  $KGH$  has been described similar to the figure  $ABC$ , and equal to  $D$ . [Q.E.F.]

## PROPOSITION 26. THEOREM.

*If two similar parallelograms have a common angle, and be similarly situated, they are about the same diagonal.*

Let the parallelograms ABCD, AEFG be similar and similarly situated, and having the common angle BAD :  
ABCD and AEFG shall be about the same diagonal.



**Construction.** For, if not, let, if possible, the parallelogram BD have its diagonal AHC in a different straight line from AF, the diagonal of the parallelogram EG ;  
let GF meet AHC at H, and draw HK parallel to AD. [I. 31.]

**Proof.** The parallelograms ABCD and AKHG are about the same diagonal, and are therefore similar ; [VI. 24.]

$$\therefore DA : AB :: GA : AK.$$

But because ABCD and AEFG are similar, [Hypothesis.]

$$\therefore DA : AB :: GA : AE ; \quad \text{[VI. Definition 2.]}$$

$$\therefore GA : AK :: GA : AE ; \quad \text{[V. 11.]}$$

$$\therefore AK = AE,$$

the less to the greater, which is impossible ;

$\therefore$  ABCD and AEFG must have their diagonals in the same straight line, that is, they are about the same diagonal.

Wherefore, *if two similar parallelograms, etc.* [Q.E.D.]

**Note.** Prop. 26 is the converse of Prop. 24.

## EXERCISES.

1. In the figure of VI. 24 shew that EG and KH are parallel.
2. Prove also that the parallelograms EG, BF, FD, and HK are proportionals.
3. Construct a square equal to a given equilateral triangle.
4. Construct an equilateral triangle equal to a given square.
5. Construct an equilateral triangle equal to a given hexagon.
6. Through a given point draw a chord in a given circle so that it shall be divided at the point in a given ratio.

[The rect. contained by the segments is known ; for it is equal to the rect. contained by the segments of any chord through the given point. Thus, the ratio of the segments being given, we have to construct a rectangle of given area, whose sides are in a given ratio, so that the required rectangle is to be similar to a given one.]

## PROPOSITION 30. PROBLEM.

*To cut a given straight line in extreme and mean ratio.*

Let AB be the given straight line :  
*it is required to cut it in extreme and mean ratio.*

A                      C                      B

**Construction.** Divide AB at the point C, so that the rectangle AB, BC may be equal to the square on AC. [II. 11.]

**Proof.** Because the rectangle AB, BC

= the square on AC,

[Construction.]

$\therefore AB : AC :: AC : CB.$

[VI. 17.]

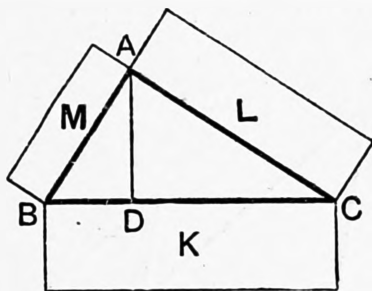
Wherefore AB is cut in extreme and mean ratio at the point C.

[Q.E.F. VI. Definition 4.]

## PROPOSITION 31. THEOREM.

*In any right-angled triangle, any rectilineal figure described on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.*

Let  $ABC$  be a triangle, right-angled at  $A$  :  
*the rectilineal figure described on  $BC$  shall be equal to the similar and similarly described figures on  $BA$  and  $CA$ .*



**Construction.** Draw the perpendicular  $AD$ . [I. 12.]

Let  $K$ ,  $L$ ,  $M$  be the similar and similarly described figures on  $BC$ ,  $CA$ , and  $AB$ .

**Proof.** Because  $AD$  is drawn from the right angle  $A$  perpendicular to  $BC$ ,

the  $\triangle CBA$  is similar to the  $\triangle ABD$ ; [VI. 8.]

$\therefore BC : BA :: BA : BD$ ; [VI. Def. 2.]

$\therefore BC : BD :: K : M$ ; [VI. 20, Cor. 2.]

and inversely,  $BD : BC :: M : K$ . [V. B.]

Similarly,  $CD : BC :: L : K$ ;

$\therefore$  as  $BD$  and  $CD$  together are to  $BC$  so are  $L$  and  $M$  together to  $K$ . [V. 24.]

But  $BD$  and  $CD$  together  $= BC$ ;

$\therefore$  the figure  $K =$  the figures  $L$  and  $M$ . [V. A.]

Wherefore, *in any right-angled triangle, etc.* [Q.E.D.]

*Note.* Of this general proposition I. 47 is a particular case.

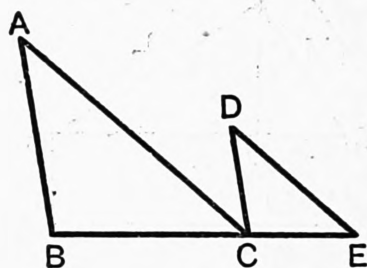


## PROPOSITION 32. THEOREM.

*If two triangles, which have two sides of one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel, the remaining sides shall be in a straight line.*

Let  $ABC$  and  $DCE$  be two triangles which have  $BA$  to  $AC$  as  $CD$  is to  $DE$ , and let  $AB$  be parallel to  $DC$  and  $AC$  parallel to  $DE$ :

$BC$  and  $CE$  shall be in one straight line.



**Proof.** Because  $AC$  meets the parallels  $AB$ ,  $DC$ , [Hyp.  
the alternate angles  $BAC$ ,  $ACD$  are equal; [I. 29.  
Similarly, the angles  $ACD$ ,  $CDE$  are equal;  
 $\therefore$  the  $\angle BAC =$  the  $\angle CDE$ . [Axiom 1.

Again, because the  $\angle$  at  $A =$  the  $\angle$  at  $D$ ,  
and that  $BA : AC :: CD : DE$ , [Hypothesis.  
 $\therefore$  the  $\triangle ABC$  is equiangular to the  $\triangle DCE$ ; [VI. 6.

$\therefore$  the  $\angle ABC = \angle DCE$ .

Also the  $\angle BAC = \angle ACD$ ; [Shewn.  
 $\therefore$  the whole  $\angle ACE =$  the two angles  $ABC$  and  $BAC$ . [Ax. 2.

Add the  $\angle ACB$  to each of these equals;

$\therefore \angle^s ACE, ACB$  together  $= \angle^s ABC, BAC, ACB$ ,

that is,  $=$  two right angles;

$\therefore BC$  and  $CE$  are in one straight line.

Wherefore, if two triangles, etc.

[I. 32.

[I. 14.

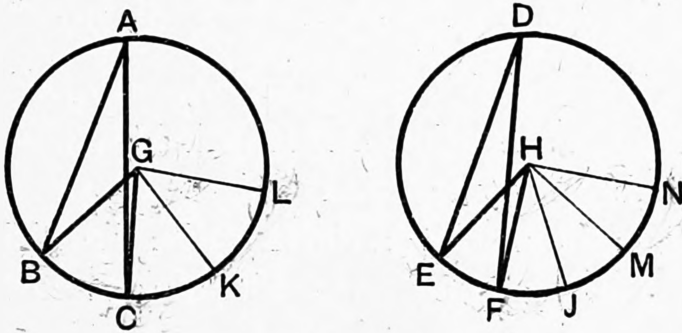
[Q. E. D.

PROPOSITION 33. THEOREM.

*In equal circles, angles, whether at the centres or at the circumferences, have the same ratio which the arcs on which they stand have to one another ; so also have the sectors.*

Let  $ABC$  and  $DEF$  be equal circles, and let  $BGC$  and  $EHF$  be angles at their centres, and  $BAC$  and  $EDF$  angles at their circumferences : then shall

- (i.) *the  $\angle BGC$  be to the  $\angle EHF$  as the arc  $BC$  to the arc  $EF$  ;*
- (ii.) *the  $\angle BAC$  be to the  $\angle EDF$  as the arc  $BC$  to the arc  $EF$  ;*
- and (iii.) *the sector  $BGC$  be to the sector  $EHF$  as the arc  $BC$  to the arc  $EF$ .*



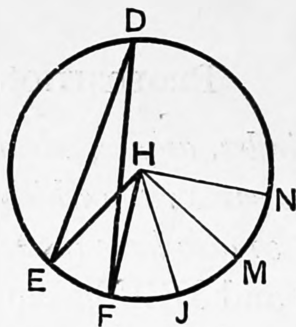
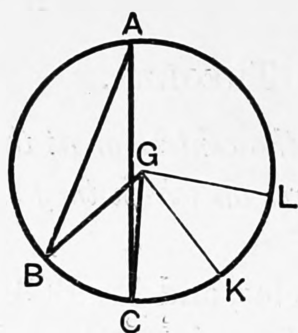
**Construction.** Take any number of arcs  $CK$ ,  $KL$ , each equal to  $BC$ , and also any number of arcs  $FJ$ ,  $JM$ ,  $MN$  each equal to  $EF$  ; and join  $GK$ ,  $GL$ ,  $HJ$ ,  $HM$ ,  $HN$ .

**Proof.** (i.) Because the arcs  $BC$ ,  $CK$ ,  $KL$  are all equal, the angles  $BGC$ ,  $CGK$ ,  $KGL$  are also all equal ; [III. 27.  $\therefore$  the arc  $BL$  is the same multiple of the arc  $BC$  that the  $\angle BGL$  is of the  $\angle BGC$ .

Similarly, the arc  $EN$  is the same multiple of the arc  $EF$  that the  $\angle EHN$  is of the  $\angle EHF$ .

And if the arc  $BL =$  the arc  $EN$ , the  $\angle BGL =$  the  $\angle EHN$  ; [III. 27. and if the arc  $BL$  be greater than the arc  $EN$ , the  $\angle BGL$  is greater than the  $\angle EHN$  ; and if less, then less.

Therefore, since there are four magnitudes, the two arcs  $BC$ ,  $EF$ , and the two angles  $BGC$ ,  $EHF$  ;



and of the first and third have been taken any equimultiples whatever, namely, the arc BL and the  $\angle BGL$ ;

and of the second and fourth have been taken any equimultiples whatever, namely, the arc EN and the  $\angle EHN$ ;

and since the  $\angle BGL$  is  $>$ ,  $=$ , or  $<$  the  $\angle EHN$ , according as the arc BL is  $>$ ,  $=$ , or  $<$  the arc EN;

$\therefore$  arc BC : arc EF ::  $\angle BGC$  :  $\angle EHF$ .

[V. Definition 5.]

(ii.) But  $\angle BGC$  :  $\angle EHF$  ::  $\angle BAC$  :  $\angle EDF$ ,

[V. 15.]

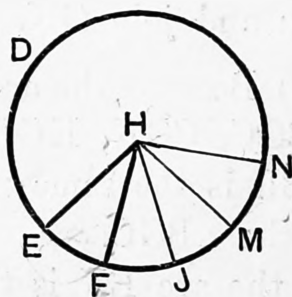
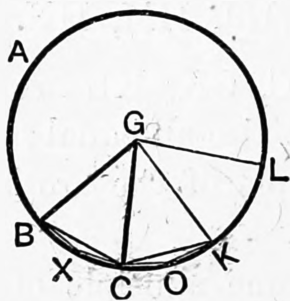
for each is double of each;

[III. 20.]

$\therefore$  the arc BC : the arc EF :: the  $\angle BGC$  : the  $\angle EHF$ ,

and :: the  $\angle BAC$  : the  $\angle EDF$ .

(iii.) Join BC, CK, and in the arcs BC, CK take any points X, O, and join BX, XC, CO, OK.



Then, in the triangles BGC, CGK,

because  $\left\{ \begin{array}{l} BG = GC, \\ GC = CK, \\ \text{and } \angle BGC = \angle CGK; \end{array} \right.$

$\therefore$  the base BC = the base CK, and  $\triangle BGC = \triangle CGK$ . [I. 4.]

Also because the arc  $BC =$  the arc  $CK$ , [Construction.  
 $\therefore$  the remaining arc  $BAC =$  the remaining arc  $CAK$ ;  
 $\therefore$  the  $\angle BXC =$  the  $\angle COK$ ; [III. 27.  
 $\therefore$  the segments  $BXC$ ,  $COK$  are similar, [III. Definition 11.  
 and they are on equal chords  $BC$ ,  $CK$ ;  
 $\therefore$  the segment  $BXC =$  the segment  $COK$ . [III. 24.

Also the  $\triangle BGC$  was shewn to be equal to the  $\triangle CGK$ ;  
 $\therefore$  the whole sector  $BGC =$  the whole sector  $CGK$ . [Axiom 2.  
 Similarly, the sector  $KGL =$  each of the sectors  $BGC$ ,  $CGK$ .  
 In the same manner the sectors  $EHF$ ,  $FHJ$ ,  $JHM$ ,  $MHN$   
 may be shewn to be equal,  
 $\therefore$  whatever multiple the arc  $BL$  is of the arc  $BC$ , the same  
 multiple is the sector  $BGL$  of the sector  $BGC$ ;  
 and, similarly, whatever multiple the arc  $EN$  is of the arc  $EF$ ,  
 the same multiple is the sector  $EHN$  of the sector  $EHF$ .  
 Also if arc  $BL =$  the arc  $EN$ , the sector  $BGL =$  the sector  $EHN$ ;  
 and if the arc  $BL$  be greater than the arc  $EN$ , the sector  $BGL$   
 is greater than the sector  $EHN$ ; and if less, then less.

Therefore, since there are four magnitudes, the two arcs  $BC$ ,  
 $EF$ , and the two sectors  $BGC$ ,  $EHF$ ;  
 and that of the first and third have been taken any equi-  
 multiples whatever, namely, the arc  $BL$  and the sector  $BGL$ ;  
 and of the second and fourth have been taken any equimultiples  
 whatever, namely, the arc  $EN$  and the sector  $EHN$ ;  
 and since the sector  $BGL$  is  $>$ ,  $=$ , or  $<$  the sector  $EHN$ ,  
 according as the arc  $BL$  is  $>$ ,  $=$ , or  $<$  the arc  $EN$ ;

$\therefore$  the arc  $BC$  : the arc  $EF$

:: the sector  $BGC$  : sector  $EHF$ . [V. Definition 5.

Wherefore, if equal circles, etc.

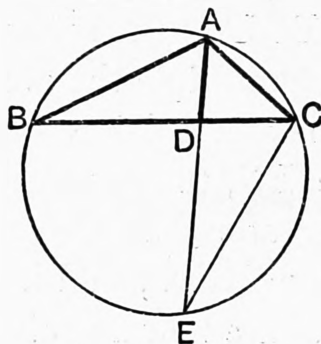
[Q.E.D.

*Note.* In VI. 33 Euclid implicitly gives up the restriction, which  
 he seems to have adopted hitherto, that no angle is to be considered  
 greater than two right angles. For in the demonstration the angle  
 $BGL$  may be any multiple whatever of the angle  $BGC$ , and so may be  
 greater than any number of right angles.

## PROPOSITION B. THEOREM.

*If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.*

Let  $ABC$  be a triangle, and let the angle  $BAC$  be bisected by  $AD$ ;  
*the rectangle  $BA, AC$  shall be equal to the rectangle  $BD, DC$ , together with the square on  $AD$ .*



**Construction.** Describe the circle  $ACB$  about the triangle;  
 [IV. 5.]  
 produce  $AD$  to meet the circumference at  $E$ , and join  $EC$ .

**Proof.** Because the  $\angle BAD =$  the  $\angle EAC$ , [Hypothesis.  
 and the  $\angle ABD =$  the  $\angle AEC$ , in the same segment. [III. 21.]

$\therefore$  the  $\triangle BAD$  is equiangular to the  $\triangle EAC$ ;

$\therefore BA : AD :: EA : AC$ ; [VI. 4.]

$\therefore$  the rect.  $BA, AC =$  the rect.  $EA, AD$ , [VI. 16.]  
 that is,  $=$  rect.  $ED, DA$ , and the square on  $AD$  [II. 3.]

But the rect.  $ED, DA =$  the rect.  $BD, DC$ ; [III. 35.]

$\therefore$  the rect.  $BA, AC =$  the rect.  $BD, DC$ , together with  
 the square on  $AD$ .

Wherefore, *if the vertical angle, etc.*

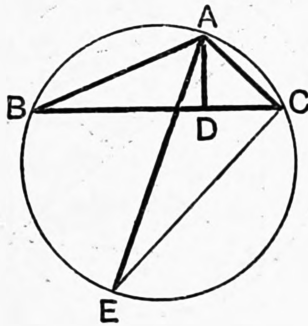
[Q.E.D.]

## PROPOSITION C. THEOREM.

*If from the vertical angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*

Let  $ABC$  be a triangle, and let  $AD$  be the perpendicular from  $A$  to the base  $BC$ :

*the rect.  $BA, AC$  shall be equal to the rect. contained by  $AD$  and the diameter of the circle described about the triangle.*



**Construction.** Describe the circle  $ACB$  about the triangle; [IV. 5.]  
draw the diameter  $AE$ , and join  $EC$ .

**Proof.** Because the right angle  $BDA$   
= the  $\angle ECA$  in a semi-circle; [III. 31.]  
and the  $\angle ABD =$  the  $\angle AEC$  in the same segment; [III. 21.]  
 $\therefore$  the  $\triangle ABD$  is equiangular to the  $\triangle AEC$ ;  
 $\therefore BA : AD :: EA : AC$ ; [VI. 4.]  
 $\therefore$  the rect.  $BA, AC =$  the rect.  $EA, AD$ . [VI. 16.]  
Wherefore, *if from the vertical angle, etc.* [Q.E.D.]

## EXERCISES ON PROPOSITIONS B, C.

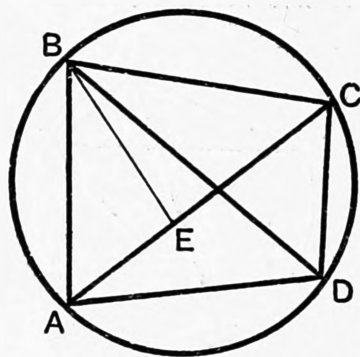
**1.** If the exterior angle of  $BAC$  of a triangle  $ABC$  be bisected by a straight line  $AE$  which cuts the base in  $F$ , and the circle in  $E$ , prove that the rectangle  $BF, FC =$  the rectangle  $BA, AC +$  the square on  $AF$ .  
[The  $\triangle^s BAF, EAC$  are similar.]

**2.** Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

PROPOSITION D. THEOREM. ✓

*The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.*

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD:  
*the rect. AC, BD shall be equal to the two rectangles contained by AB, CD and by AD, BC.*



**Construction.** Make the  $\angle ABE$  equal to the  $\angle DBC$ . [I. 23.]

**Proof.** Add to each of these equals the  $\angle EBD$ ;  
 then the  $\angle ABD =$  the  $\angle EBC$ . [Axiom 2.]

Also the  $\angle BDA =$  the  $\angle BCE$ , in the same segment; [III. 21.]

$\therefore$  the  $\triangle ABD$  is equiangular to the  $\triangle EBC$ ;

$\therefore AD : DB :: EC : CB$ ; [VI. 4.]

$\therefore$  the rect. AD, CB = the rect. DB, EC. [VI. 16.]

Again, because the  $\angle ABE =$  the  $\angle DBC$ , [Construction.]  
 and the  $\angle BAE =$  the  $\angle BDC$  in the same segment; [III. 21.]

$\therefore$  the  $\triangle ABE$  is equiangular to the  $\triangle DBC$ ;

$\therefore BA : AE :: BD : DC$ ; [VI. 4.]

$\therefore$  the rect. BA, DC = the rect. AE, BD. [VI. 16.]

But the rect. AD, CB = the rect. DB, EC; [Shewn.]

$\therefore$  the rectangles AD, CB and BA, DC together  
 = the rectangles BD, EC and BD, AE;

that is, = the rectangle BD, AC. [II. 1.]

Wherefore, *the rectangle contained, etc.*

[Q. E. D.]



**EXERCISES.**

**1.** *The sum of the rectangles contained by opposite sides of a quadrilateral is greater than that contained by the diagonals except when the quadrilateral can be inscribed in a circle, and then it is equal to the rectangle contained by the diagonals.*

Let ABCD be a quadrilateral not inscribable in a circle.

Make the  $\angle ABE = \angle DBC$ , and the  $\angle BAE = \angle BDC$ .

[Since ABCD is not inscribable in a circle, the  $\angle BAC$  does not  $= \angle BDC$ , and  $\therefore$  AE does not fall on AC as it does in Proposition D.]

The  $\triangle^s$  ABE, DBC are thus equiangular;  $\therefore AB : AE :: DB : DC$ .

$$\therefore AB \cdot DC = AE \cdot DB.$$

Again, since  $\angle ABD = \angle EBC$ , and that

$AB : BE :: BD : BC$ , since  $\triangle^s$  ABE, DBC are similar,

$$i.e. AB : BD :: BE : BC.$$

$\therefore \triangle^s$  ABD, EBC are similar, and

$$\therefore AD : DB :: CE : CB, i.e. AD \cdot CB = CE \cdot DB$$

$$\therefore AB \cdot DC + AD \cdot CB = AE \cdot DB + CE \cdot DB$$

= rectangle contained by DB and the sum of AE, EC,

and  $\therefore >$  rectangle contained by DB, AC.

[I. 20.]

**2.** A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle; shew that one of these straight lines is equal to the sum of the other two.

**3.** From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn at right angles to AB, AC respectively, and intersecting at D; shew that the rectangle BC, AD is double of the rectangle AB, DB.

**4.** If the diagonals of a quadrilateral inscribed in a circle be at right angles, the sum of the rectangles contained by opposite sides is equal to twice the area of the quadrilateral.

**5.** Prove that the product of the perpendiculars drawn from any point on a circle upon a pair of opposite sides of an inscribed quadrilateral is equal to the product of the perpendiculars from the same point on the other pair of sides, and also to the product of the perpendiculars upon the diagonals of the quadrilateral.

[If O be the point, and OF, OG, OH, OK, OL, OM the perpendiculars on the sides AB, BC, CD, DA and the diagonals AC, BD of the quadrilateral ABCD, then, by Prop. C,

$$OF \times \text{diam}^r = OA \cdot OB; OH \times \text{diam}^r = OC \cdot OD,$$

$$OL \times \text{diam}^r = OA \cdot OC; OM \times \text{diam}^r = OB \cdot OD;$$

$$\therefore OF \cdot OH = OL \cdot OM. \text{ So } OG \cdot OK = OL \cdot OM.]$$



## BOOK XI.

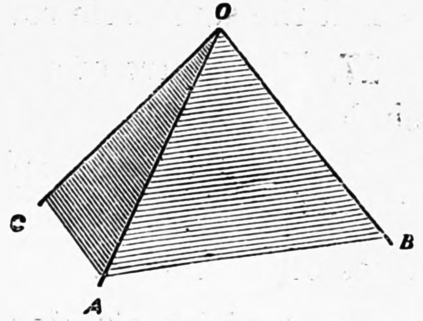
### DEFINITIONS.

1. A **solid** is that which has length, breadth, and thickness.
2. That which bounds a solid is a superficies.
3. A straight line is **perpendicular, or at right angles, to a plane**, when it is at right angles to every straight line meeting it in that plane. Such a straight line is often called a **normal** to the plane.
4. A plane is **perpendicular to a plane** when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.
5. The **inclination of a straight line to a plane** is the acute angle contained by that straight line, and another drawn from the point at which the first line meets the plane to the point at which a perpendicular to the plane drawn from any point of the first line above the plane meets the same plane.
6. The **inclination of a plane to a plane** is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one in one plane, and the other in the other plane.
7. Two planes are said to have the same or a like inclination to one another which two other planes have, when the said angles of inclination are equal to one another.
8. **Parallel planes** are such as do not meet one another though produced.

A straight line is said to be parallel to a plane when they do not meet if produced.

9. A **solid angle** is that which is made by more than two plane angles, which are not in the same plane, meeting at one point.

A solid angle contained by three plane angles is called a **trihedral angle**; one contained by more than three plane angles is called a **polyhedral angle**.



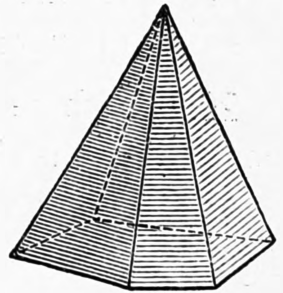
10. The angle made by two straight lines, which do not meet, is the angle contained by two straight lines parallel to them and drawn through any point.

11. Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

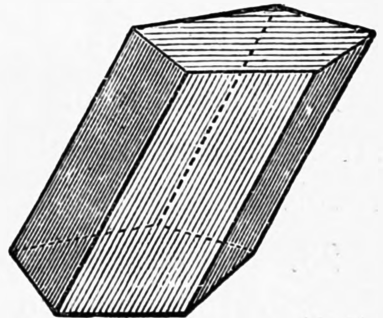
12. A **polyhedron** is a solid figure all of whose bounding surfaces are planes.

It is said to be **regular** when its bounding surfaces are equal and regular polygons.

13. A **pyramid** is a polyhedron which has for one face a triangle or polygon, and for the other faces triangles whose bases are the sides of the polygon, and a point which does not lie in the plane of the polygon as the common vertex.



14. A **prism** is a polyhedron contained by plane figures, of which two that are opposite are equal, similar and parallel to one another, and the others are parallelograms.



15. A **sphere** is a solid figure described by the revolution of a semicircle about its diameter, which remains fixed.

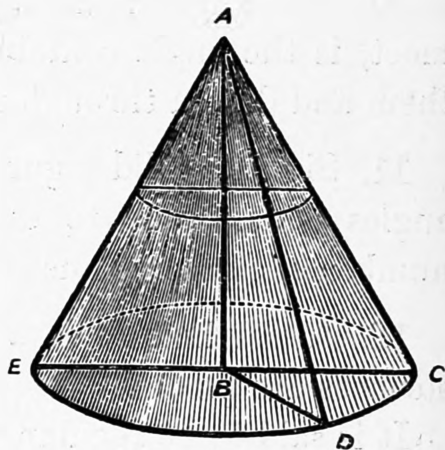
16. The axis of a sphere is the fixed straight line about which the semicircle revolves.

17. The centre of a sphere is the same as that of the semicircle.

The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

18. A **right circular cone** is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled cone; and if greater, an acute-angled cone.



19. The axis of a cone is the fixed straight line about which the triangle revolves.

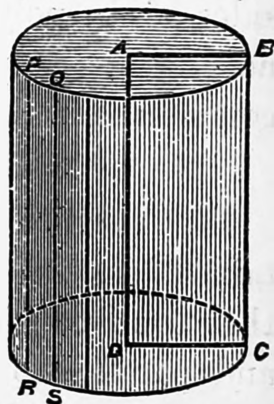
20. The base of a cone is the circle described by that side containing the right angle which revolves.

21. A **right circular cylinder** is a solid figure described by the revolution of a rectangle about one of its sides which remains fixed.

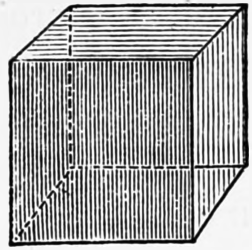
22. The axis of a cylinder is the fixed straight line about which the rectangle revolves.

23. The bases of a cylinder are the circles described by the two revolving opposite sides of the rectangle.

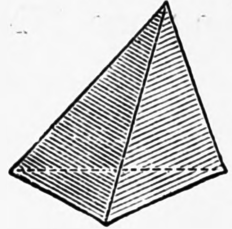
24. Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.



25. A **cube** is a solid figure contained by six equal squares.



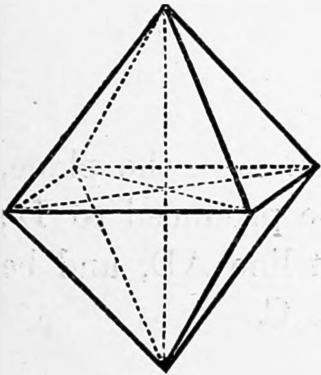
26. A **tetrahedron** is a solid figure contained by four triangles; when these bounding triangles are all equal and equilateral, the tetrahedron is said to be **regular**.



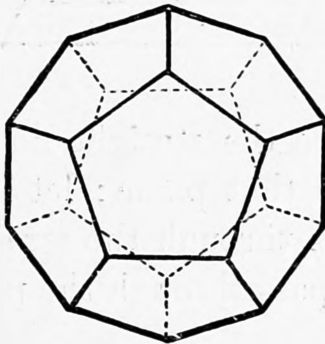
27. A regular **octahedron** is a solid figure contained by eight equal and equilateral triangles.

28. A regular **dodecahedron** is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

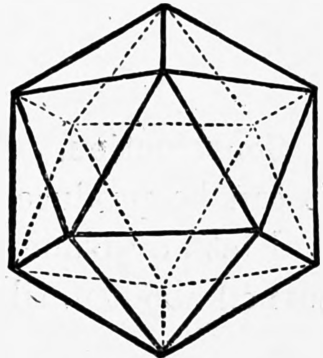
29. A regular **icosahedron** is a solid figure contained by twenty equal and equilateral triangles.



OCTAHEDRON.

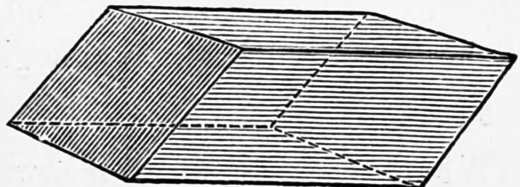


DODECAHEDRON.

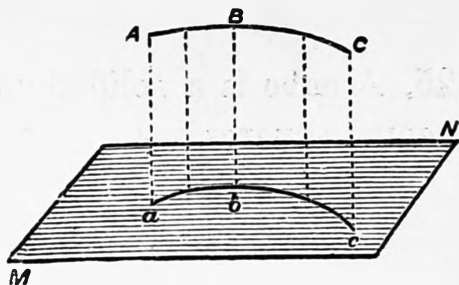


ICOSAHDREDON.

30. A **parallelepiped** is a solid figure contained by six quadrilateral figures, of which every opposite two are parallel.



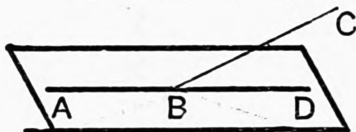
**31.** The projection of a line on a plane is the line joining the feet of the perpendiculars let fall from the points in that line upon the plane. Thus  $abc$  is the projection of  $ABC$  upon the plane  $MN$ .



### PROPOSITION 1 THEOREM.

*One part of a straight line cannot be in a plane, and another part without it.*

If it be possible, let  $AB$ , part of the straight line  $ABC$ , be in a plane, and the part  $BC$  without it.



**Construction.** Since the straight line  $AB$  is in the plane, it can be produced in that plane; let it be produced to  $D$ ; and let any plane pass through the straight line  $AD$ , and be turned about until it pass through the point  $C$ .

**Proof.** Because the points  $B$  and  $C$  are in this plane, the straight line  $BC$  is in it. [I. Definition 7.]

Therefore there are two straight lines  $ABC$ ,  $ABD$  in the same plane that have a common segment  $AB$ ;

but this is impossible.

[I. Axiom 10.]

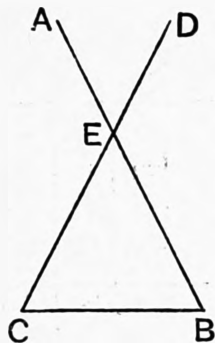
Wherefore, *one part of a straight line, etc.*

[Q. E. D.]

## PROPOSITION 2. THEOREM.

*Two straight lines which cut one another are in one plane ; and three straight lines which meet one another are in one plane.*

Let the two straight lines  $AB$ ,  $CD$  cut one another at  $E$  :  
 $AB$  and  $CD$  shall be in one plane ; and the three straight lines  $EC$ ,  $CB$ ,  $BE$ , which meet one another, shall be in one plane.



**Construction.** Let any plane pass through the straight line  $EB$ , and let the plane be turned about  $EB$ , produced if necessary, until it pass through the point  $C$ .

**Proof.** Because the points  $E$  and  $C$  are in this plane, the straight line  $EC$  is in it ; [I. Definition 7.

for the same reason, the straight line  $BC$  is in the same plane ; and, by hypothesis,  $EB$  is in it ;

$\therefore EC$ ,  $CB$ ,  $BE$  are in one plane.

But  $AB$ ,  $CD$  are in the plane in which  $EB$ ,  $EC$  are ; [XI. 1.

$\therefore AB$  and  $CD$  are in one plane.

Wherefore, *two straight lines, etc.*

[Q.E.D.]

**Note.** A plane is determined when it is given that it passes through

(1) two straight lines, intersecting or parallel ; [Cf. XI. 7.

or (2) three points which are not in a straight line ;

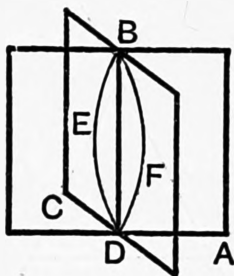
or (3) a given straight line and a given point not lying on this straight line.

## PROPOSITION 3. THEOREM.

*If two planes cut one another, their common section is a straight line.*

Let two planes AB, BC cut one another, and let BD be their common section :

*BD shall be a straight line.*



**Construction.** If it be not, from B to D, draw in the plane AB the straight line BED, and in the plane BC the straight line BFD. [Postulate 1.]

**Proof.** The two straight lines BED, BFD have the same extremities, and therefore include a space between them ;

but this is impossible. [Axiom 10.]

Therefore BD, the common section of the planes AB and BC, cannot but be a straight line.

Wherefore, *if two planes, etc.*

[Q.E.D.]

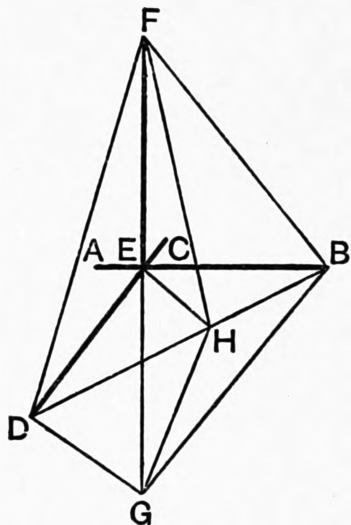


## PROPOSITION 4. THEOREM.

*If a straight line stand at right angles to each of two straight lines at the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.*

Let the straight line  $EF$  stand at right angles to each of the straight lines  $AB$ ,  $CD$  at  $E$ , the point of their intersection:  $EF$  shall also be at right angles to the plane passing through  $AB$ ,  $CD$ .

**Construction.** Produce  $FE$  to  $G$ , making  $EG$  equal to  $FE$ ; through  $E$  draw any straight line  $EH$ , meeting  $BD$  in  $H$ , and join  $FD$ ,  $FH$ ,  $FB$ ,  $GD$ ,  $GH$ ,  $GB$ .



[Construction.]

**Proof.** (1) In the triangles  $BEF$ ,  $BEG$ ,  
 because  $\begin{cases} FE = EG, \\ \text{and } EB \text{ is common,} \\ \text{and the } \angle BEF = \text{the } \angle BEG, \text{ both being right angles;} \end{cases}$   
 $\therefore BF = BG.$  [I. 4.]

Similarly,  $DF = DG.$   
 The triangles  $FDB$ ,  $GDB$  are therefore equal in all respects,  
 and therefore the  $\angle FBD = \text{the } \angle GBD.$

(2) In the triangles  $FBH$ ,  $GBH$ ,  
 because  $\begin{cases} FB = BG, \\ \text{and } BH \text{ is common,} \\ \text{and the } \angle FBD = \text{the } \angle GBD; \end{cases}$  [Proved.]  
 $\therefore \text{the base } FH = \text{the base } GH.$  [Proved.]

(3) Lastly, in the triangles  $FEH$ ,  $GEH$ ,  
 because  $\begin{cases} FE = GE, \\ \text{and } EH \text{ is common,} \\ \text{and the base } FH = \text{the base } GH; \end{cases}$  [Construction.]  
 $\therefore \text{the } \angle GEH = \text{the } \angle FEH;$  [Proved in (2).]  
 $\therefore \text{each is a right } \angle;$  [I. Def. 10.]  
 $\therefore EF \text{ is perpendicular to } EH.$

Similarly, it may be shewn to be  $\perp^r$  to every other straight line which meets it in the plane through  $AB$  and  $CD$ ;  
 $\therefore EF \text{ is } \perp^r \text{ to the plane in which are } AB \text{ and } CD.$  [XI. Def. 3.]

Wherefore, *if a straight line, etc.*

[Q.E.D.]

**Note.** This is not the proof as given by Euclid, but is much easier.



## EXERCISES.

**\*\*1.** The angle between two planes is equal to the angle between two perpendiculars, one to one plane and one to the other, which intersect.

**2.** Prove that three planes, in general, meet in a point. What are the exceptions?

**\*\*3.** Of all the straight lines that can be drawn from a given point to a given plane, the least is that which is perpendicular to the given plane; and of other such lines that one, whose intersection with the plane is nearer to the foot of the perpendicular, is less than one whose intersection is farther from this foot.

**\*\*4.** The sum of the squares on any two opposite edges of a tetrahedron is less than the sum of the squares on the other four edges.

[Let ABCD be the tetrahedron, E and F the middle points of AB and CD. Then by Ex. 1, page 109, we have

$$AC^2 + AD^2 = 2AF^2 + 2FD^2 \text{ and } BC^2 + BD^2 = 2BF^2 + 2FD^2.$$

$$\begin{aligned} \therefore AC^2 + AD^2 + BC^2 + BD^2 &= 2AF^2 + 2BF^2 + 4FD^2 \\ &= 4FE^2 + 4BE^2 + 4FD^2, \text{ by the same theorem,} \\ &= 4FE^2 + AB^2 + CD^2, \end{aligned}$$

that is, the sum of the squares on AB, CD is less than the sum of the squares on the other four edges by four times the square on the line joining the middle points of AB and CD.]

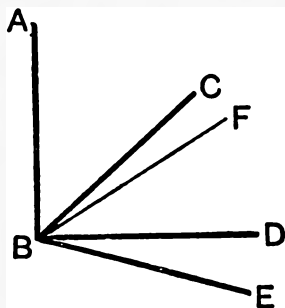
**\*\*5.** In a tetrahedron the sum of the squares on the six edges is always equal to four times the sum of the squares on the straight lines joining the middle points of opposite edges. [Use Ex. 4.]

## PROPOSITION 5. THEOREM.

*If three straight lines meet all at one point, and a straight line be perpendicular to each of them at that point, the three straight lines shall be in one and the same plane.*

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, at B the point where they meet :

BC, BD, BE shall be in one and the same plane.



**Construction.** For, if not, let, if possible, BD and BE be in one plane, and BC without it; let a plane pass through AB and BC; the common section of this plane with the plane in which are BD and BE is a straight line; [XI. 3. let this straight line be BF.

**Proof.** The three straight lines AB, BC, BF are all in one plane, namely, the plane which passes through AB and BC.

Because AB is perpendicular to both BD and BE, [Hyp.  $\therefore$  it is perpendicular to the plane through them; [XI. 4.  $\therefore$  it is perpendicular to the straight line BF which meets it, and is in that plane; [XI. Definition 3.

$\therefore$  the  $\angle ABF$  is a right  $\angle$ .

But the  $\angle ABC$  is, by hypothesis, a right  $\angle$ ;

$\therefore$  the  $\angle ABC =$  the  $\angle ABF$ , [Axiom 11.

and they are in one plane, which is impossible; [Axiom 8.

$\therefore$  the straight line BC is not without the plane in which are BD and BE,

that is, BC, BD, BE are in one and the same plane.

Wherefore, if three straight lines, etc.

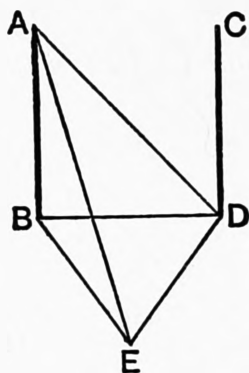
[Q.E.D.

## PROPOSITION 6. THEOREM.

*If two straight lines be at right angles to the same plane, they shall be parallel to one another.*

Let the straight lines  $AB$ ,  $CD$  be at right angles to the same plane :

$AB$  shall be parallel to  $CD$ .



**Construction.** Let them meet the plane at  $B$ ,  $D$  ; join  $BD$  ; and in the plane draw  $DE$  at right angles to  $BD$  ; [I. 11. make  $DE$  equal to  $AB$  ; and join  $BE$ ,  $AE$ ,  $AD$ .

**Proof.** Because  $AB$  is perpendicular to the plane, [Hyp. it is at right angles to  $BD$  and  $BE$ , which meet it and are in that plane ; [XI. Definition 3.

$\therefore$  each of the angles  $ABD$ ,  $ABE$  is a right angle.

Similarly, each of the angles  $CDB$ ,  $CDE$  is a right angle.

Also, because  $ABE$  is a right  $\angle$ ,

$\therefore$  sq. on  $AE$  = sum of the sqs. on  $AB$ ,  $BE$  [I. 47.

= sum of the sqs. on  $AB$ ,  $BD$ ,  $DE$ , since  $BDE$  is a right  $\angle$ ,

= sum of the squares on  $AD$ ,  $DE$ , since  $ADB$  is a rt.  $\angle$  ;

$\therefore$  the  $\angle ADE$  is a right angle. [I. 48.

But the angles  $EDB$ ,  $EDC$  are also right angles ;

$\therefore$   $DB$ ,  $DA$ ,  $DC$  are all in the same plane. [XI. 5.

But  $AB$  is in the plane in which are  $BD$ ,  $DA$  ; [XI. 2.

$\therefore$   $AB$ ,  $BD$ ,  $CD$  are in one plane.

Also each of the angles  $ABD$ ,  $CDB$  is a right  $\angle$  ;

$\therefore$   $AB$  is parallel to  $CD$ . [I. 28.

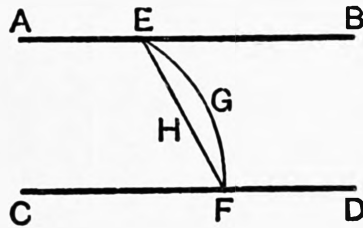
Wherefore, if two straight lines, etc.

[Q.E.D.

## PROPOSITION 7. THEOREM.

*If two straight lines be parallel, the straight line drawn from any point in one to any point in the other is in the same plane with the parallels.*

Let  $AB$ ,  $CD$  be parallel straight lines, and take any point  $E$  in one and any point  $F$  in the other :  
*the straight line which joins  $E$  and  $F$  shall be in the same plane with the parallels  $AB$  and  $CD$ .*



For, if not, let it be, if possible, without the plane, as  $EGF$ ; and in the plane  $ABCD$ , in which the parallels are, draw the straight line  $EHF$  from  $E$  to  $F$ .

Then, since  $EGF$  is also a straight line, [Hypothesis.  
the two straight lines  $EGF$ ,  $EHF$  include a space between them, which is impossible; [Axiom 10.

$\therefore$  the straight line joining the points  $E$  and  $F$  is not without the plane in which the parallels  $AB$ ,  $CD$  are;

$\therefore$  it is in that plane.

Wherefore, *if two straight lines, etc.*

[Q.E.D.]

## EXERCISES.

**\*\*1.** If a straight line be parallel to a second straight line it is parallel to any plane passing through the second straight line.

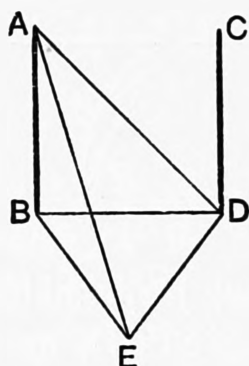
**\*\*2.** If the perpendiculars drawn from two points to a plane be equal, the straight line joining the two points is parallel to the plane.

## PROPOSITION 8. THEOREM.

*If two straight lines be parallel, and one of them be at right angles to a plane, the other also shall be at right angles to the same plane.*

Let  $AB$ ,  $CD$  be two parallel straight lines ; and let one of them  $AB$  be at right angles to a plane :

*the other  $CD$  shall be at right angles to the same plane.*



**Construction.** Let  $AB$ ,  $CD$  meet the plane at the points  $B$ ,  $D$  ; join  $BD$  ; then  $AB$ ,  $CD$ ,  $BD$  are in one plane. [XI. 7. In the plane to which  $AB$  is at right angles, draw  $DE$  at right angles to  $BD$  ; [I. 11. make  $DE$  equal to  $AB$  ; and join  $BE$ ,  $AE$ ,  $AD$ .

**Proof.** (1) Because  $AB$  is at right angles to the plane, [*Hyp.* it is at right angles to every straight line meeting it in that plane ; [XI. *Definition* 3.

$\therefore$  each of the angles  $ABD$ ,  $ABE$  is a right angle.

And because  $BD$  meets the two parallels  $AB$ ,  $CD$ , the angles  $ABD$ ,  $CDB$  together = two right angles. [I. 29.

But the  $\angle ABD$  is a right  $\angle$  ; [*Hypothesis.*

$\therefore$  the  $\angle CDB$  is a right  $\angle$ ,

that is,  $CD$  is at right angles to  $BD$ .

(2) Because  $\angle ABE$  is a right  $\angle$ ;

$\therefore$  the sq. on  $AE$  = the squares on  $AB$ ,  $BE$  [I. 47.

= the squares on  $AB$ ,  $BD$ ,  $DE$ , since  $\angle BDE$  is a right  $\angle$ ,

= the squares on  $AD$ ,  $DE$ , since  $\angle ABD$  is a right  $\angle$ ;

$\therefore$  the  $\angle ADE$  is a right angle, [I. 48.

that is,  $ED$  is at right angles to  $AD$ .

But  $ED$  is at right angles to  $BD$ ; [Construction.

$\therefore$   $ED$  is at right angles to the plane which passes through  $BD$ ,  $DA$ . [XI. 4.

But  $CD$  is in the plane passing through  $BD$ ,  $DA$ , because all three are in the plane in which are the parallels  $AB$ ,  $CD$ ;

$\therefore$   $ED$  is at right angles to  $CD$ ; [XI. Def. 3.

$\therefore$   $CD$  is at right angles to  $ED$ .

But  $CD$  was shewn to be also at right angles to  $BD$ ;

$\therefore$   $CD$  is at right angles to the plane passing through  $BD$ ,  $ED$ , [XI. 4.

that is, to the plane to which  $AB$  is at right angles.

Wherefore, *if two straight lines, etc.* [Q.E.D.

### EXERCISE.

From a point  $A$  a perpendicular is drawn to a plane meeting it at  $B$ ; from  $B$  a perpendicular is drawn to a straight line  $DE$  in the plane meeting it at  $C$ : shew that  $AC$  is perpendicular to the straight line  $DE$ .

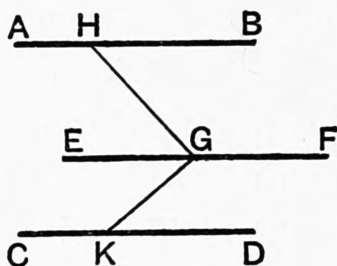
[Through  $B$  draw  $BK \parallel$  to  $DE$ ; then  $BK$  is in the plane and  $\angle ABK$  is a right  $\angle$  since  $AB$  is  $\perp^r$  to the plane;  $\therefore$   $\angle KBA$ ,  $\angle KBC$  are right  $\angle^s$ ;  $\therefore$   $KB$  is  $\perp^r$  to the plane through  $BC$ ,  $BA$ ;  $\therefore$  the  $\parallel$  line  $DE$  is  $\perp^r$  to the plane through  $BC$ ,  $CA$ ;  $\therefore$   $\angle DCA$  is a right  $\angle$ , etc.]

## PROPOSITION 9. THEOREM.

*Two straight lines which are each of them parallel to the same straight line, and are not in the same plane with it, are parallel to one another.*

Let  $AB$  and  $CD$  be each of them parallel to  $EF$ , and not in the same plane with it:

$AB$  shall be parallel to  $CD$ .



**Construction.** In  $EF$  take any point  $G$ ;  
in the plane passing through  $EF$  and  $AB$ , draw from  $G$  the straight line  $GH$  at right angles to  $EF$ ;  
and in the plane passing through  $EF$  and  $CD$ , draw from  $G$  the straight line  $GK$  at right angles to  $EF$ . [I. 11.]

**Proof.** Because  $EF$  is at right angles to  $GH$  and  $GK$ , [*Constr.*]  
 $EF$  is at right angles to the plane  $HGK$  passing through them. [XI. 4.]

Also  $EF$  is parallel to  $AB$  and also to  $CD$ ; [*Hypothesis.*]  
 $\therefore AB$  and  $CD$  are each of them at right angles to the plane  $HGK$ ; [XI. 8.]

$\therefore AB$  is parallel to  $CD$ . [XI. 6.]

Wherefore, if two straight lines, etc. [Q.E.D.]

## EXERCISE.

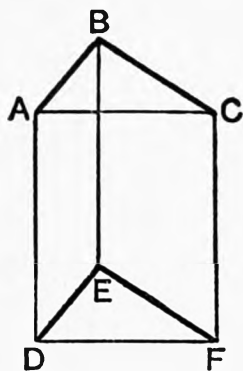
\*\*The middle points of two pairs of opposite edges of a tetrahedron are in one plane and form a parallelogram.

[Let  $ABCD$  be the tetrahedron;  $P$  and  $Q$  the middle points of the opposite edges  $AB$ ,  $CD$ , and  $R$ ,  $S$  the middle points of the other edges  $AD$ ,  $BC$ . Then  $QR$  is  $\parallel^1$  to and half of  $AC$ ; so  $PS$  is  $\parallel^1$  to and half of  $AC$ ;  $\therefore QR$  and  $PS$  are  $\parallel^1$  and equal;  $\therefore$  etc.]

## PROPOSITION 10. THEOREM.

*If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two, the first two and the other two shall contain equal angles.*

Let the two straight lines AB, BC, which meet one another, be parallel to the two straight lines DE, EF, which meet one another, and are not in the same plane with AB, BC:  
*the angle ABC shall be equal to the angle DEF.*



**Construction.** Take BA, BC, ED, EF all equal to one another, and join AD, BE, CF, AC, DF.

**Proof.** Because AB is equal and parallel to DE,  
 $\therefore$  AD is equal and parallel to BE. [I. 33.]

Similarly, CF is equal and parallel to BE;

$\therefore$  AD and CF are each of them equal and parallel to BE;

$\therefore$  AD is equal and parallel to CF; [Axiom 1 and XI. 9.]

$\therefore$  AC is equal and parallel to DF. [I. 33.]

Also because AB, BC are equal to DE, EF, each to each,  
 and the base AC = the base DF;

$\therefore$  the  $\angle ABC =$  the  $\angle DEF$ . [I. 8.]

Wherefore, if two straight lines, etc.

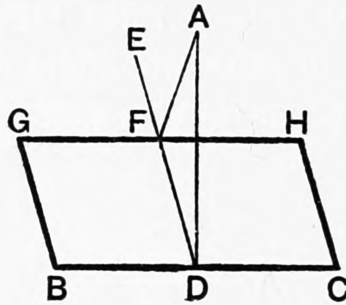
[Q.E.D.]



## PROPOSITION 11. PROBLEM.

*To draw a straight line perpendicular to a given plane from a given point without it.*

Let  $A$  be the given point without the plane  $BH$  :  
*it is required to draw from  $A$  a straight line perpendicular to the plane  $BH$ .*



**Construction.** Draw any straight line  $BC$  in the plane  $BH$ , and from  $A$  draw  $AD$  perpendicular to  $BC$ . [I. 12.]  
 Then, if  $AD$  be also perpendicular to the plane  $BH$ , the thing required is done. But, if not, from  $D$  draw, in the plane  $BH$ , the straight line  $DE$  at right angles to  $BC$ , [I. 11.]  
 and from  $A$  draw  $AF$  perpendicular to  $DE$  : [I. 12.]  
 *$AF$  shall be perpendicular to the plane  $BH$ .*

Through  $F$  draw  $GH$  parallel to  $BC$ . [I. 31.]

**Proof.** Because  $BC$  is at right angles to  $ED$  and  $DA$ , [*Constr.*]  $BC$  is at right angles to the plane through  $ED$ ,  $DA$ . [XI. 4.]  
 Also  $GH$  is parallel to  $BC$ ; [*Construction.*]  
 $\therefore GH$  is at right angles to the plane passing through  $ED$  and  $DA$ . [XI. 8.]

But  $AF$  meets  $GH$ , and is in this plane ;

$\therefore GH$  is at right angles to  $AF$  ; [XI. Definition 3.]

$\therefore AF$  is at right angles to  $GH$ .

But  $AF$  is also at right angles to  $DE$ ; [*Construction.*]

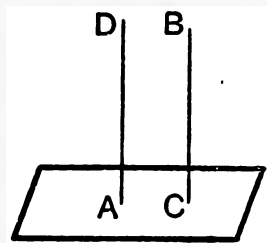
$\therefore AF$  is perpendicular to the plane through  $GH$  and  $DE$ ,  
 that is, to the plane  $BH$ . [XI. 4.]

Wherefore, *from the given point  $A$ , without the plane  $BH$ , the straight line  $AF$  has been drawn perpendicular to the plane.* [Q.E.F.]

## PROPOSITION 12. PROBLEM.

*To draw a straight line perpendicular to a given plane, from a given point in the plane.*

Let  $A$  be the given point in the plane :  
*it is required to draw from  $A$  a straight line perpendicular to the plane.*



**Construction.** From any point  $B$  without the plane, draw  $BC$  perpendicular to the plane ; [XI. 11.  
 and from  $A$  draw  $AD$  parallel to  $BC$  : [I. 31.  
 *$AD$  shall be the straight line required.*

**Proof.** Because  $AD$  and  $BC$  are two parallel straight lines, [Construction.  
 and that one of them  $BC$  is perpendicular to the given plane, [Construction.  
 the other  $AD$  is also perpendicular to the given plane. [XI. 9.

Wherefore a straight line has been drawn perpendicular to a given plane, from a given point in it. [Q. E. F.

## EXERCISES.

**\*\*1.** Shew that equal straight lines drawn from a given point to a given plane are equally inclined to the plane.

**2.** If two straight lines in one plane be equally inclined to another plane, they will be equally inclined to the common section of these planes.

**\*\*3.**  $P$  and  $Q$  are two points on the same side of a given plane ; find a point  $R$  in the plane which is such that the sum of  $PR$ ,  $RQ$  is the least possible.

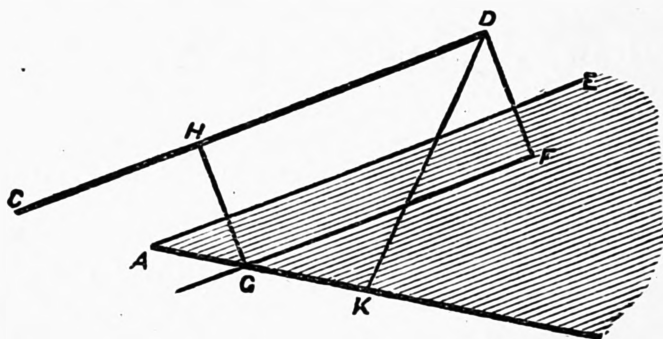
4. OA, OB, OC are three equal straight lines meeting in O which are not in the same plane ; prove that the perpendicular from O upon the plane ABC meets it in the centre of the circumcircle of the triangle ABC. [Use Ex. 1.]

\*\*5. Give a geometrical construction for drawing a straight line which shall be equally inclined to three straight lines meeting at a point. [Use Exs. 1 and 4.]

6. Perpendiculars are drawn from a point to a plane, and to a straight line in that plane ; shew that the straight line joining the feet of the perpendiculars is perpendicular to the former straight line.

7. If O is the centre of the circumcircle of a triangle ABC and OP be drawn perpendicular to the plane of the triangle, prove that any point in OP is equidistant from the vertices A, B, C.

\*\*8. Show how to construct the straight line which is perpendicular to two straight lines which are not parallel and do not intersect ; shew also that the common perpendicular is the shortest distance between the two given lines.



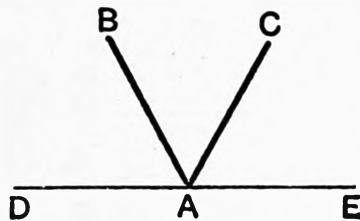
Let AB and CD be the two given straight lines ; through AB draw the plane AEB, which contains AB and a straight line AE parallel to CD ; this plane is therefore parallel to CD. Draw DF  $\perp^r$  to this plane and let the plane CDF cut the plane AEB in the straight line FG. Let FG meet AB in G and draw GH  $\perp^r$  to CD. Then GH is the line required, that is, it shall also be  $\perp^r$  to AB. For CD, being parallel to the plane AEB, does not meet GF which is in this plane. Also CD and GF are in the same plane CDF ;  $\therefore$  CD and GF are parallel ;  $\therefore$  DF, which is  $\perp^r$  to GF, is  $\perp^r$  to CD, and is  $\therefore \parallel$  to HG ;  $\therefore$  HG is  $\perp^r$  to the plane AEB, and  $\therefore$  is  $\perp^r$  to the straight line AB.

Also HG is the shortest distance between the two given lines. For let DK be any other line joining two points on them. Since DF is  $\perp^r$  to the plane EAB ;  $\therefore \angle DFK$  is a right  $\angle$  ;  $\therefore$  in the  $\triangle DFK$ ,  $DK > DF$ , i.e.  $> HG$ .

## PROPOSITION 13. THEOREM.

*From the same point in a given plane there cannot be two straight lines at right angles to the plane on the same side of it; and there can be but one perpendicular to a plane from a point without the plane.*

For, if it be possible, let the two straight lines AB, AC be at right angles to a given plane, from the same point A in the plane, and on the same side of it.



Let a plane pass through BA, AC;  
the common section of this with the given plane is a straight line;  
[XI. 3.]  
let this straight line be DAE.

**Proof.** AB, AC, DAE are all in one plane. [Const.]  
Because CA is at right angles to the given plane, [Hypothesis.]  
it is at right angles to every straight line meeting it in the plane.  
[XI. Definition 3.]

But DAE meets CA, and is in that plane;

$\therefore$  the  $\angle CAE$  is a right  $\angle$ .

For the same reason the  $\angle BAE$  is a right  $\angle$ ;

$\therefore$  the  $\angle CAE =$  the  $\angle BAE$ ,

and they are in one plane, which is impossible. [Axiom 8.]

Also, from a point without the plane, there can be but one perpendicular to the plane.

For if there could be two, they would be parallel [XI. 6.]  
which is absurd.

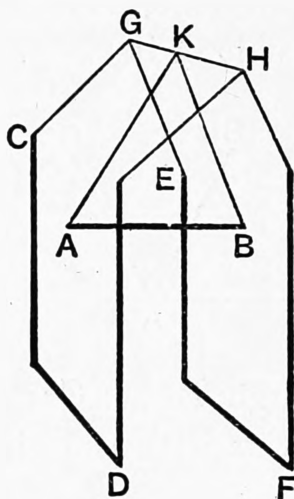
Wherefore, from the same point, etc. [Q.E.D.]

## PROPOSITION 14. THEOREM.

*Planes to which the same straight line is perpendicular are parallel to one another.*

Let the straight line  $AB$  be perpendicular to each of the planes  $CD$  and  $EF$ :

*these planes shall be parallel to one another.*



**Construction.** For, if not, they will meet when produced; let them meet; their common section will be a straight line; let  $GH$  be this straight line; in it take any point  $K$ , and join  $AK$ ,  $BK$ .

**Proof.** Because  $AB$  is perpendicular to the plane  $EF$ , [*Hyp.* it is perpendicular to the straight line  $BK$  which is in that plane; [XI. *Definition* 3.

$\therefore$  the  $\angle ABK$  is a right  $\angle$ .

For the same reason the  $\angle BAK$  is a right  $\angle$ ;

$\therefore$  the two angles  $ABK$ ,  $BAK$  of the  $\triangle ABK$  are equal to two right angles, which is impossible. [I. 17.

$\therefore$  the planes  $CD$  and  $EF$ , though produced, do not meet, that is, they are parallel. [XI. *Definition* 8.

Wherefore, *planes, etc.*

[Q. E. D.

## PROPOSITION 15. THEOREM.

*If two straight lines, which meet one another, be parallel to two other straight lines which meet one another, but are not in the same plane with the first two, the plane passing through these is parallel to the plane passing through the others.*

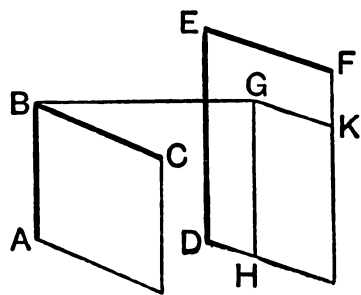
Let  $AB$ ,  $BC$ , two straight lines which meet one another, be parallel to two other straight lines  $DE$ ,  $EF$ , which meet one another, but are not in the same plane with  $AB$ ,  $BC$ :

*the plane passing through  $AB$ ,  $BC$  shall be parallel to the plane passing through  $DE$ ,  $EF$ .*

**Construction.** From  $B$  draw  $BG$  perpendicular to the plane passing through  $DE$ ,  $EF$ , [XI. 11.]

and let it meet that plane at  $G$ ;

through  $G$  draw  $GH \parallel$  to  $ED$ , and  $GK \parallel$  to  $EF$ . [I. 31.]



**Proof.** Because  $BG$  is perpendicular to the plane passing through  $DE$ ,  $EF$ , [Construction.]

it is perpendicular to each of the straight lines  $GH$  and  $GK$ , which meet it, and are in that plane; [XI. Definition 3.]

$\therefore$  each of the angles  $BGH$  and  $BGK$  is a right  $\angle$ .

Now, because  $BA$  is parallel to  $ED$ , [Hypothesis.]

and  $GH$  is parallel to  $ED$ , [Construction.]

$\therefore$   $BA$  is parallel to  $GH$ ; [XI. 9.]

$\therefore$  the angles  $ABG$  and  $BGH$  together = two right angles. [I. 29.]

Also the angle  $BGH$  has been shewn to be a right angle;

$\therefore$  the  $\angle ABG$  is a right  $\angle$ .

For the same reason the  $\angle CBG$  is a right  $\angle$ ;

$\therefore$   $GB$  is perpendicular to the plane through  $AB$ ,  $BC$ . [XI. 4.]

And  $GB$  is also  $\perp^r$  to the plane through  $DE$ ,  $EF$ ; [Constr.]

$\therefore$  the plane passing through  $AB$ ,  $BC$  is parallel to the plane passing through  $DE$ ,  $EF$ . [XI. 14.]

Wherefore, *if two straight lines, etc.*

[Q.E.D.]

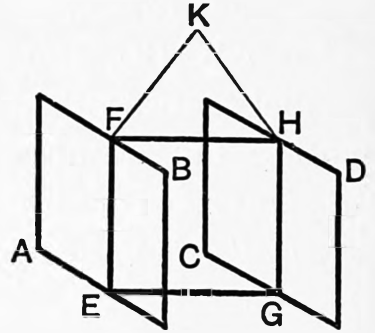
## PROPOSITION 16. THEOREM.

*If two parallel planes be cut by another plane, their common sections with it are parallel.*

Let the parallel planes  $AB$ ,  $CD$  be cut by the plane  $EFHG$ , and let their common sections with it be  $EF$ ,  $GH$ :

*$EF$  shall be parallel to  $GH$ .*

**Construction.** For if not,  $EF$  and  $GH$ , being produced, will meet either towards  $F$ ,  $H$  or towards  $E$ ,  $G$ . Let them be produced and meet towards  $F$ ,  $H$  at the point  $K$ .



**Proof.** Since  $EFK$  is in the plane  $AB$ , every point in  $EFK$  is in that plane ;

[XI. 1.]

$\therefore K$  is in the plane  $AB$ .

For the same reason  $K$  is in the plane  $CD$  ;

$\therefore$  the planes  $AB$ ,  $CD$ , being produced, meet one another.

But they do not meet, since they are parallel by hypothesis ;

$\therefore EF$  and  $GH$ , being produced, do not meet towards  $F$ ,  $H$ .

In the same manner it may be shewn that they do not meet towards  $E$ ,  $G$  ;

$\therefore EF$  is parallel to  $GH$ .

[I. Definition 29.]

Wherefore, *if two parallel planes, etc.*

[Q.E.D.]

## EXERCISES.

**1.** Draw two parallel planes, one through one straight line, and the other through another straight line which does not meet the former.

**\*\*2.** If two planes which are not parallel be cut by two parallel planes, the lines of section of the first two by the last two will contain equal angles.

**3.** Polygons formed by cutting a pyramid by parallel planes are similar.

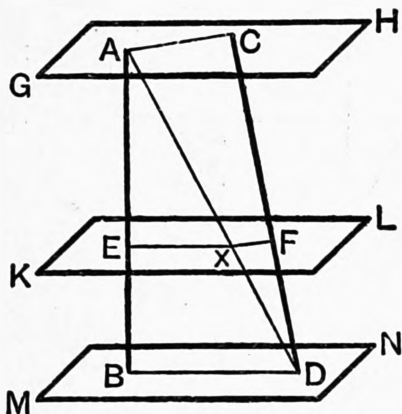
**\*\*4.** Polygons formed by cutting a prism by parallel planes are equal.

**5.** From the extremities of the two parallel straight lines  $AB$ ,  $CD$  parallel straight lines  $Aa$ ,  $Bb$ ,  $Cc$ ,  $Dd$  are drawn meeting a plane at  $a$ ,  $b$ ,  $c$ ,  $d$  : shew that  $AB$  is to  $CD$  as  $ab$  is to  $cd$ .

## PROPOSITION 17. THEOREM.

*If two straight lines be cut by parallel planes, they shall be cut in the same ratio.*

Let the straight lines AB and CD be cut by the parallel planes GH, KL, MN at the points A, E, B and C, F, D : AE shall be to EB as CF to FD.



**Construction.** Join AC, BD, AD ; let AD meet the plane KL at the point X ; and join EX, XF.

**Proof.** Because the parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are parallel ; [XI. 16.

and because the parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel. [XI. 16.

Also because EX is parallel to BD, a side of the  $\triangle ABD$ ,

$$\therefore AE : EB :: AX : XD. \quad [\text{VI. 2.}]$$

Again, because XF is parallel to AC, a side of the  $\triangle ADC$ ,

$$\therefore AX : XD :: CF : FD ; \quad [\text{VI. 2.}]$$

$$\therefore AE : EB :: CF : FD. \quad [\text{V. 11.}]$$

Wherefore, *if two straight lines, etc.*

[Q.E.D.]



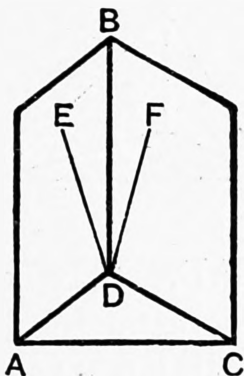


## PROPOSITION 19. THEOREM.

*If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.*

Let the two planes BA, BC be each of them perpendicular to a third plane, and let BD be the common section of the planes BA, BC:

*BD shall be perpendicular to the third plane*



**Construction.** For, if not, from the point D, draw in the plane BA the straight line DE at right angles to AD, the common section of the plane BA with the third plane; [I. 11. and from D, draw in the plane BC the straight line DF at right angles to CD, the common section of the plane BC with the third plane. [I. 11.

**Proof.** Because the plane BA is perpendicular to the third plane, and DE is drawn in the plane BA at right angles to AD their common section; [Construction.

$\therefore$  DE is perpendicular to the third plane. [XI. Definition 4.

Similarly, DF is perpendicular to the third plane;  
 $\therefore$  from D two straight lines are drawn at right angles to the third plane, on the same side of it, which is impossible; [XI. 13.  
 $\therefore$  from D there cannot be drawn any straight line at right angles to the third plane, except BD the common section of the planes BA, BC;

$\therefore$  BD is perpendicular to the third plane.

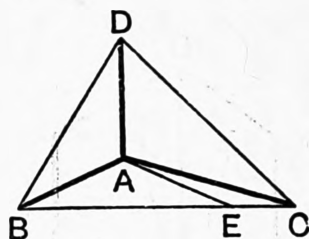
Wherefore, *if two planes, etc.*

[Q.E.D.]

## PROPOSITION 20. THEOREM.

*If a solid angle be contained by three plane angles, any two of them are together greater than the third.*

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB :  
*any two of them shall be together greater than the third.*



If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third.

If they are not all equal, let BAC be that angle which is not less than either of the other two, and is greater than one of them, BAD.

**Construction.** At the point A, in the straight line BA, make, in the plane which passes through BA, AC, the angle BAE equal to the angle BAD ;

[I. 23.]

make AE equal to AD ;

[I. 3.]

through E draw BEC, cutting AB, AC at the points B, C, and join DB, DC.

**Proof.** In the triangles BAD, BAE,

because  $\begin{cases} AD = AE, \\ \text{and } AB \text{ is common,} \\ \text{and the } \angle BAD = \text{the } \angle BAE, \end{cases}$   
 $\therefore$  the base BD = the base BE.

[Construction.]

[Construction.]

[I. 4.]

Also because BD, DC are together greater than BC, [I. 20. and one of them BD has been shewn equal to BE, a part of BC ;

$\therefore$  the other, DC, is greater than the remaining part EC.

Also because  $AD = AE$ , [Construction.]  
 and  $AC$  is common to the two  $\triangle^s$   $DAC$ ,  $EAC$ ,  
 but the base  $DC >$  the base  $EC$ ;

$\therefore$  the  $\angle DAC >$  the  $\angle EAC$ . [I. 25.]

Also, by construction, the  $\angle BAD =$  the  $\angle BAE$ ;  
 $\therefore$  the angles  $BAD$ ,  $DAC$  are together greater than the  
 angles  $BAE$ ,  $EAC$ , that is, than the angle  $BAC$ .

Again, the  $\angle BAC$  is not less than either of the angles  
 $BAD$ ,  $DAC$ ;

$\therefore$  the  $\angle BAC$  together with either of the other angles is  
 greater than the third.

Wherefore, *if a solid angle, etc.* [Q.E.D.]

### EXERCISES ON PROPOSITIONS 18 and 19.

**1.** From a point  $A$  in one of two planes are drawn  $AB$  at right angles to the first plane, and  $AC$  perpendicular to the second plane, and meeting the second plane at  $B$ ,  $C$ ; shew that  $BC$  is perpendicular to the line of intersection of the two planes.

[By Prop. 18 the plane  $ABC$  is perpendicular to each of the given planes, and  $\therefore$  by Prop. 19 is at right angles to their intersection;  $\therefore$  their intersection is  $\perp^r$  to  $BC$  which is in the plane  $ABC$ .]

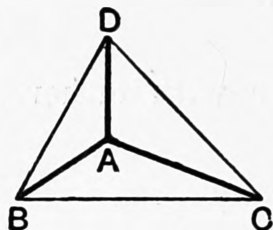
**2.** Perpendiculars  $AE$ ,  $BF$  are drawn to a plane from two points  $A$ ,  $B$  above it; a plane is drawn through  $A$  perpendicular to  $AB$ ; shew that its line of intersection with the given plane is perpendicular to  $EF$ .

[Let  $EF$  meet the line of intersection  $OC$  in  $O$ . Then the plane  $AOE$  is  $\perp^r$  to the plane  $AOC$ , since it goes through  $AB$  the perpendicular to  $AOC$  (XI. 18). Similarly it is  $\perp^r$  to the plane  $EOC$ ;  $\therefore$  it is  $\perp^r$  to their intersection  $OC$ ;  $\therefore$   $OC$  is  $\perp^r$  to the plane  $AOE$ , and  $\therefore$  to  $OE$ .]

## PROPOSITION 21. THEOREM.

*Every solid angle is contained by plane angles, which are together less than four right angles.*

(i.) Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB :  
*these three shall be together less than four right angles.*



**Construction.** In the straight lines AB, AC, AD take any points B, C, D, and join BC, CD, DB.

**Proof.** Because the solid angle at B is contained by the three plane angles CBA, ABD, DBC ;

the angles CBA, ABD are together  $>$  the  $\angle$  DBC. [XI. 20.  
 Similarly, the  $\angle^s$  BCA, ACD are together  $>$  the  $\angle$  DCB.

and the angles CDA, ADB are together  $>$  the  $\angle$  BDC ;

the six angles CBA, ABD, BCA, ACD, CDA, ADB are together  $>$  the three angles DBC, DCB, BDC,  
 that is,  $>$  two right angles. [I. 32.

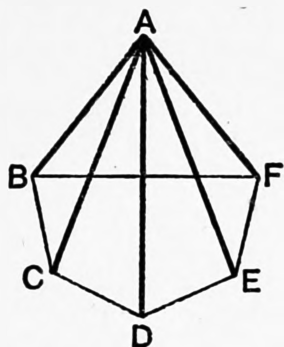
Also because the three angles of each of the triangles ABC, ACD, ADB together = two right angles, [I. 32.  
 therefore the nine angles of these triangles, namely, the angles CBA, BCA, BAC, ACD, CDA, CAD, ADB, ABD, DAB = six right angles ;

and of these, the six angles CBA, BCA, ACD, CDA, ADB, ABD are greater than two right angles ;

$\therefore$  the remaining three angles BAC, CAD, DAB, which contain the solid angle at A, are together  $<$  four right  $\angle^s$ .

(ii.) Let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB :  
*these shall be together less than four right angles.*

**Construction.** Let the planes, in which the angles are, be cut by a plane, and let the common sections of it with these planes be BC, CD, DE, EF, FB.



**Proof.** Because the solid angle at B is contained by the three plane angles CBA, ABF, FBC ;

$\therefore$  the angles CBA, ABF are together  $>$  the  $\angle$  FBC. [XI. 20.

Similarly, at each of the points C, D, E, F, the two plane angles which are at the bases of the triangles having the common vertex A, are together  $>$  the third  $\angle$  at the same point, which is one of the angles of the polygon BCDEF ;  
 $\therefore$  all the angles at the bases of the triangles are together greater than all the angles of the polygon.

Now all the angles of the triangles together  
 = twice as many right angles as there are triangles,  
 that is, as there are sides in the polygon BCDEF ; [I. 32.  
 and all the  $\angle^s$  of the polygon, together with four right  $\angle^s$ , also  
 = twice as many right angles as there are sides in the  
 polygon ; [I. 32, Corollary 1.

$\therefore$  all the angles of the triangles = all the angles of the polygon, together with four right angles. [Axiom 1.

But it has been shewn that all the angles at the bases of the triangles are together greater than all the angles of the polygon ;

$\therefore$  the remaining  $\angle^s$  of the triangles, namely, those at the vertex, which contain the solid angle at A are  $<$  four right angles.

Wherefore, *every solid angle, etc.*

[Q. E. D.

*Note 1.* In the second case of the foregoing proposition it is assumed that the polygon BCDEF has no *re-entrant* angle. [See note to I. 32.] Otherwise the proposition will not be true.

*Note 2.* There cannot be more than five *regular* polyhedra.

For we require not less than three plane angles to form a solid angle. Also by the preceding proposition the angles forming any solid angle are together less than four right angles.

Now three times the angle of a regular hexagon equals four right angles; and three times the angle of a polygon, with a number of sides larger than six, is greater than four right angles. The faces of a regular polyhedron must thus have less than six sides each, and must therefore be triangles, squares, or pentagons.

It is therefore only possible that a solid angle of a regular polyhedron should be formed by

- (1) three equilateral triangles,
- (2) four equilateral triangles,
- (3) five equilateral triangles,
- (4) three squares, or
- (5) three pentagons.

For any higher number such as six equilateral triangles, or four squares, or four pentagons, would make the plane angles at each solid angle together not less than four right angles.

The solids corresponding to the above five cases are

- (1) a tetrahedron formed by four equilateral triangles,
- (2) an octahedron formed by eight equilateral triangles,
- (3) an icosahedron formed by twenty equilateral triangles,
- (4) a cube formed by six squares, and
- (5) a dodecahedron formed by twelve pentagons.

There are no other *regular* polyhedra besides these five.

## BOOK XII.

### LEMMA.

[This lemma is the First Proposition of the Tenth Book.]

*If from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the smaller of the proposed magnitudes.*

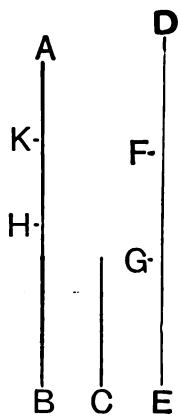
Let AB and C be two unequal magnitudes, of which AB is the greater :

*if from AB there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than C.*

For C may be multiplied so as to become greater than AB.

Let it be so multiplied, and let DE its multiple be greater than AB, and be divided into DF, FG, GE, each equal to C.

From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and so on, until there be as many divisions in AB as in DE ; and let the divisions in AB be AK, KH, HB.



Then, because  $DE > AB$ ,

and that EG taken from DE is not  $>$  its half,

but BH taken from AB  $>$  its half ;

$\therefore$  the remainder DG  $>$  the remainder AH.

Again, because DG  $>$  AH, and that

GF is not  $>$  the half of DG, but HK  $>$  the half of AH ;

$\therefore$  the remainder DF  $>$  the remainder AK.

But  $DF = C$  ;  $\therefore C > AK$  ; that is,  $AK < C$ .

[Q.E.D.]

Also if only the halves be taken away, the same thing may in the same way be demonstrated.

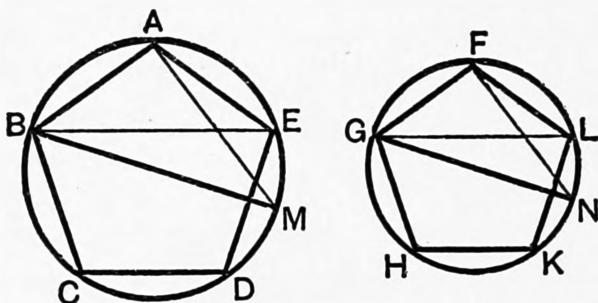


## PROPOSITION 1. THEOREM.

*Similar polygons inscribed in circles are to one another as the squares on their diameters.*

Let  $ABCDE$ ,  $FGHKL$  be two circles, and in them the similar polygons  $ABCDE$ ,  $FGHKL$ ; and let  $BM$ ,  $GN$  be the diameters of the circles :

*the polygon  $ABCDE$  shall be to the polygon  $FGHKL$  as the square on  $BM$  is to the square on  $GN$ .*



**Construction.** Join  $AM$ ,  $BE$ ,  $FN$ ,  $GL$ .

**Proof.** Because the polygons are similar

$\therefore$  the  $\angle BAE =$  the  $\angle GFL$ ,

and  $BA : AE :: GF : FL$ ;

[VI. Definition 2.

$\therefore$  the  $\triangle BAE$  is equiangular to the  $\triangle GFL$ ;

[VI. 6.

$\therefore$  the  $\angle AEB =$  the  $\angle FLG$ .

But  $\angle AEB = \angle AMB$ , and  $\angle FLG = \angle FNG$ ; [III. 21.

$\therefore$  the  $\angle AMB =$  the  $\angle FNG$ .

Also, the  $\angle BAM =$  the  $\angle GFN$ ; for each is a right  $\angle$ ; [III. 31.  
 $\therefore$  the remaining angles in the triangles  $AMB$ ,  $FNG$  are equal, and the triangles are equiangular;

$\therefore BA : BM :: GF : GN$ , [VI. 4.

and, alternately,  $BA : GF :: BM : GN$ ; [V. 16.

But the polygon  $ABCDE$  is to the polygon  $FGHKL$

in the duplicate ratio of  $BA$  to  $GF$ , [VI. 20.

that is, in the duplicate ratio of  $BM$  to  $GN$ ,

that is, in the ratio of the sq. on  $BM$  to the sq. on  $GN$ . [VI. 20.

Wherefore, *similar polygons, etc.*

[Q.E.D.

## PROPOSITION 2. THEOREM.

*Circles are to one another as the squares on their diameters.*

Let ABCD, EFGH be two circles, and BD, FH their diameters :

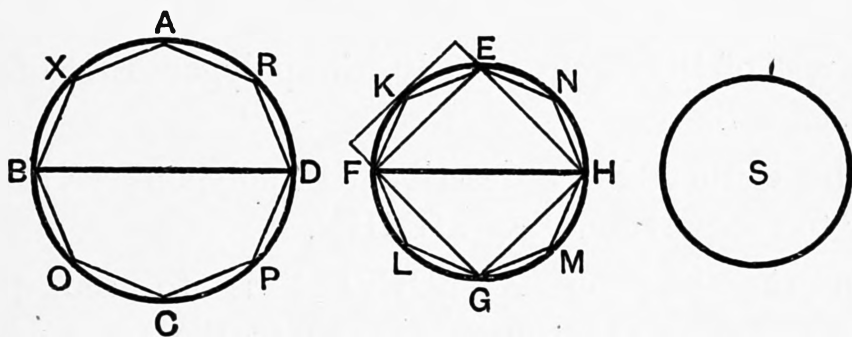
*the circle ABCD shall be to the circle EFGH as the square on BD is to the square on FH.*

For, if not, the square on BD must be to the square on FH as the circle ABCD is to some space either less than the circle EFGH, or greater than it.

*First, if possible, let it be as the circle ABCD is to a space S less than the circle EFGH.*

In the circle EFGH inscribe the square EFGH. [IV. 6.

This square shall be  $>$  half of the circle EFGH.



For the square EFGH is half of the square which can be formed by drawing straight lines to touch the circle at the points E, F, G, H ;

and the square thus formed  $>$  the circle ;

$\therefore$  the square EFGH  $>$  half of the circle.

Bisect the arcs EF, FG, GH, HE at the points K, L, M, N ; and join EK, KF, FL, LG, GM, MH, HN, NE. Then each of the triangles EKF, FLG, GMH, HNE shall be  $>$  half of the segment of the circle in which it stands.

For the triangle EKF is half of the parallelogram which can be formed by drawing a straight line to touch the circle at K, and parallels through E and F, and the parallelogram thus formed  $>$  the segment FEK ;

$\therefore$  the  $\triangle EKF >$  half of the segment.

Also similarly for the other triangles ;

$\therefore$  the sum of all these triangles together  $>$  half of the sum of the segments of the circle in which they stand.

Again, bisect  $EK$ ,  $KF$ , etc., and form triangles as before ; then the sum of these triangles  $>$  half of the sum of the segments of the circle in which they stand.

If this process be continued, and the triangles be supposed to be taken away, there will at length remain segments of circles which are together  $<$  the excess of the circle  $EFGH$  above the space  $S$ , by the preceding Lemma.

Let then the segments  $EK$ ,  $KF$ ,  $FL$ ,  $LG$ ,  $GM$ ,  $MH$ ,  $HN$ ,  $NE$  be those which remain, and which are together  $<$  the excess of the circle above  $S$  ;

$\therefore$  the rest of the circle, namely, the polygon  $EKFLGMHN$ ,  $>$  the space  $S$ .

In the circle  $ABCD$  describe the polygon  $AXBOCPDR$  similar to the polygon  $EKFLGMHN$ .

Then the polygon  $AXBOCPDR$  is to the polygon  $EKFLGMHN$  as the square on  $BD$  to the square on  $FH$ ,  
[XII. 1.]

that is, as the circle  $ABCD$  is to the space  $S$ . [*Hyp.*, V. 11.]

But the polygon  $AXBOCPDR <$  the circle  $ABCD$  in which it is inscribed ;

$\therefore$  the polygon  $EKFLGMHN <$  the space  $S$  ; [V. 14.]

but it is also greater, as has been shewn, which is impossible ;

$\therefore$  the square on  $BD$  is not to the square on  $FH$  as the circle  $ABCD$  is to any space less than the circle  $EFGH$ .

In the same way it may be shewn that the square on  $FH$  is not to the square on  $BD$  as the circle  $EFGH$  is to any space less than the circle  $ABCD$ .

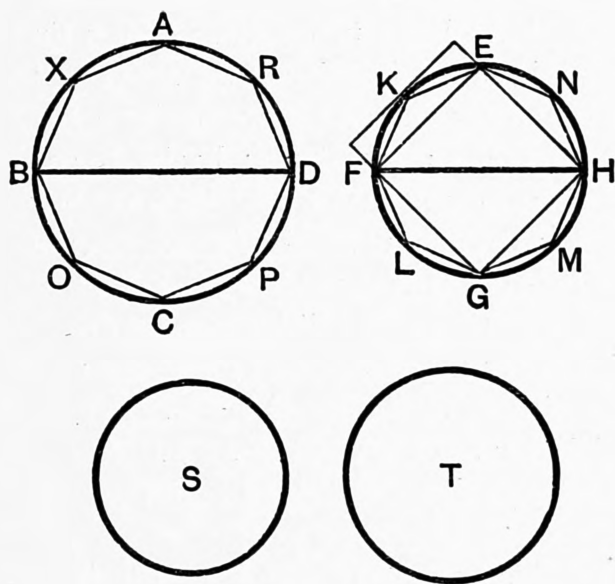
*Next*, if possible, let the square on BD be to the square on FH as the circle ABCD is to any space **greater than** the circle EFGH.

For, if possible, let it be as the circle ABCD is to a space T greater than the circle EFGH.

Then, inversely, the square on FH is to the square on BD as the space T is to the circle ABCD.

But as the space T is to the circle ABCD, so is the circle EFGH to some space, which must be less than the circle ABCD, because, by hypothesis, the space T is greater than the circle EFGH ;

[V. 14.]



$\therefore$  the square on FH is to the square on BD as the circle EFGH is to some space less than the circle ABCD, which has been shewn to be impossible ;

$\therefore$  the square on BD is not to the square on FH as the circle ABCD is to any space greater than the circle EFGH.

Also it has been shewn that the square on BD is not to the square on FH as the circle ABCD is to any space less than the circle EFGH ;

$\therefore$  the square on BD is to the square on FH as the circle ABCD is to the circle EFGH.

Wherefore, *circles, etc.*

[Q.E.D.]

## NOTES ON EUCLID'S ELEMENTS.

THE article "Eucleides," in Dr. Smith's *Dictionary of Greek and Roman Biography*, was written by Professor De Morgan; it contains an account of the works of Euclid, and of the various editions of them which have been published. To that article we refer the student who desires full information on these subjects. Perhaps the only work of importance relating to Euclid which has been published since the date of that article is a work on the *Porisms of Euclid*, by Chasles. Paris, 1860.

Euclid appears to have lived in the time of the first Ptolemy, B.C. 323–283, and to have been the founder of the Alexandrian mathematical school. The work on Geometry known as *The Elements of Euclid* consists of thirteen books; two other books have sometimes been added, of which it is supposed that Hypsicles was the author. Besides the *Elements*, Euclid was the author of other works, some of which have been preserved and some lost.

We will now mention the three editions which are the most valuable for those who wish to read the *Elements of Euclid* in the original Greek.

(1) The Oxford edition in folio, published in 1703, by David Gregory, under the title *Εὐκλείδου τὰ σωζόμενα*. "As an edition of the whole of Euclid's works, this stands alone, there being no other in Greek."—*De Morgan*.

(2) *Euclidis Elementorum Libri sex priores*, edidit Joannes Gulielmus Camerer. This edition was published at Berlin in two volumes octavo, the first volume in 1824 and the second in 1825. It contains the first six books of the *Elements* in Greek, with a Latin translation, and very good notes, which form a mathematical commentary on the subject.

(3) *Euclidis Elementa ex optimis libris in usum tironum Græce, edita ab Ernesto Ferdinando August*. This edition was published at Berlin in two volumes octavo, the first volume in 1826 and the second in 1829. It contains the thirteen books of the *Elements* in Greek, with a collection of various readings. A third volume, which was to have contained the remaining works of Euclid, never appeared. "To the scholar who wants one edition of the *Elements* we should decidedly recommend this, as bringing together all that has been done for the text of Euclid's greatest work."—*De Morgan*.

An edition, in five volumes, of the whole of Euclid's works in the original has been issued by Teubner, the well-known German publisher, as one of his series of compact editions of Greek and Latin authors.

Robert Simson's edition of the *Elements of Euclid*, which we have in substance adopted in the present work, differs considerably from the original. The English reader may ascertain the contents of the original by consulting the work entitled *The Elements of Euclid with dissertations*, by James Williamson. This work consists of two volumes quarto; the first volume was published at Oxford in 1781, and the second at London in 1788. Williamson gives a close translation of the thirteen books of the *Elements* into English, and he indicates by the use of italics the words which are not in the original, but which are required by our language.

For the history of Geometry the student is referred to Montucla's *Histoire des Mathématiques*, and to Chasles's *Aperçu historique sur l'origine et le développement des Méthodes en Géométrie*.

## BOOK I.

*Definitions.* The first seven definitions have given rise to considerable discussion, on which however we do not propose to enter. Such a discussion would consist mainly of two subjects, both of which are unsuitable to an elementary work, namely, an examination of the origin and nature of some of our elementary ideas, and a comparison of the original text of Euclid with the substitutions for it proposed by Simson and other editors. For the former subject the student may hereafter consult Whewell's *History of Scientific Ideas* and Mill's *Logic*, and for the latter the notes in Camerer's edition of the *Elements of Euclid*.

We will only observe that the ideas which correspond to the words *point*, *line*, and *surface*, do not admit of such definitions as will really supply the ideas to a person who is destitute of them. The so-called definitions may be regarded as cautions or restrictions.

Thus a *point* is not to be supposed to have any *size*, but only *position*; a *line* is not to be supposed to have any *breadth* or *thickness*, but only *length*;

a *surface* is not to be supposed to have any *thickness*, but only *length* and *breadth*.

The eighth definition seems intended to include the cases in which an angle is formed by the meeting of two *curved* lines, or of a *straight* line and a *curved* line; this definition however is of no importance, as the only angles ever considered are such as are formed by straight lines.

Some writers object to such definitions as those of an equilateral triangle, or of a square, in which the existence of the object defined is *assumed* when it ought to be *demonstrated*. They would present them in such a form as the following: if there be a triangle having three equal sides, let it be called an equilateral triangle.

Moreover, some of the definitions are introduced prematurely. Thus, for example, take the definitions of a right-angled triangle and an obtuse-angled triangle; it is not shewn until I. 17 that a triangle cannot have both a right angle and an obtuse angle, and so cannot be at the same time right-angled and obtuse-angled. And before Axiom 11 has been given, it is conceivable that the same angle may be greater than one right angle, and less than another right angle, that is, obtuse and acute at the same time.

On the *method of superposition* we may refer to papers by Professor Kelland in the *Transactions of the Royal Society of Edinburgh*, Vols. XXI. and XXIII.

The first book is chiefly devoted to the properties of triangles and parallelograms.

We may observe that Euclid himself does not distinguish between problems and theorems except by using at the end of the investigation phrases which correspond to Q.E.F. and Q.E.D. respectively.

I. 2. This problem admits of *eight* cases in its figure. For it will be found that the given point may be found with *either* end of the given straight line; there the equilateral triangle may be described on *either* side of the straight line which is drawn, and the sides of the equilateral triangle which are produced may be produced through *either* extremity. These various cases may be left for the exercise of the student, as they present no difficulty.

There will not however always be eight different straight lines obtained which solve the problem. For example, if the point A falls on BC produced, some of the solutions obtained coincide; this depends on the fact which follows from I. 32, that the angles of all equilateral triangles are equal.

I. 5. "Join FC." Custom seems to allow this singular expression as an abbreviation for "draw the straight line FC," or for "join F to C by the straight line FC."

It has been suggested to demonstrate I. 5 by *superposition*. Conceive the isosceles triangle ABC to be taken up, and then replaced so that AB falls on the old position of AC, and AC falls on the old position of AB. Thus, in the manner of I. 4, we can shew that the angle ABC is equal to the angle ACB.



**I. 6.** This proposition is not required by Euclid before he reaches II. 4; so that I. 6 might be removed from its present place and demonstrated hereafter in other ways if we please. For example, I. 6 might be placed after I. 18, and demonstrated thus. Let the angle ABC be equal to the angle ACB; then the side AB shall be equal to the side AC. For if not, one of them must be greater than the other; suppose AB greater than AC. Then the angle ACB is greater than the angle ABC, by I. 18. But this is impossible, because the angle ACB is equal to the angle ABC, by hypothesis. Or I. 6 might be placed after I. 26, and demonstrated thus. Bisect the angle BAC by a straight line meeting the base at D. Then the triangles ABD and ACD are equal in all respects, by I. 26.

**I. 12.** Here the straight line is said to be of *unlimited* length, in order that we may ensure that it shall meet the circle.

Euclid distinguishes between the terms *at right angles* and *perpendicular*. He uses the term *at right angles* when the straight line is drawn from a point *in* another, as in I. 11; and he uses the term *perpendicular* when the straight line is drawn from a point *without* another, as in I. 12. This distinction, however, is often disregarded by modern writers.

**I. 20.** "Proclus, in his *Commentary*, relates, that the Epicureans derided Prop. 20, as being manifest even to asses, and needing no demonstration; and his answer is, that though the truth of it be manifest to our senses, yet it is science which must give the reason why two sides of a triangle are greater than the third: but the right answer to this objection, against this and the 21st, and some other plain propositions, is, that the number of axioms ought not to be increased without necessity, as it must be if these propositions be not demonstrated."—*Simson*.

**I. 21.** Here it must be carefully observed that the two straight lines are to be drawn *from the ends of the side* of the triangle. If this condition be omitted the two straight lines will not necessarily be less than two sides of the triangle.

**I. 22.** "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines A, B, C, must be greater than the third.

The condition that B and C are greater than A, ensures that the circle described from the centre G shall not fall entirely within the circle described from the centre F; the condition that A and B are



greater than  $C$ , ensures that the circle described from the centre  $F$  shall not fall entirely within the circle described from the centre  $G$ ; the condition that  $A$  and  $C$  are greater than  $B$ , ensures that one of these circles shall not fall entirely without the other. Hence the circles must meet.

**I. 26.** It will appear after I. 32 that two triangles which have two angles of the one equal to two angles of the other, each to each, have also their third angles equal. Hence we are able to include the two cases of I. 26 in one enunciation thus; *if two triangles have all the angles of the one respectively equal to all the angles of the other, each to each, and have also a side of the one, opposite to any angle, equal to the side opposite to the equal angle in the other, the triangles shall be equal in all respects.*

Besides the cases of I. 4, 8, 26 there are two other cases which will naturally occur to a student to consider, namely,

(1) when two triangles have the three angles of the one respectively equal to the three angles of the other;

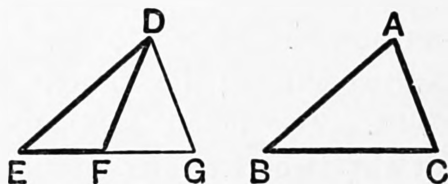
(2) when two triangles have two sides of the one equal to two sides of the other, each to each, and an angle opposite to one side of one triangle equal to the angle opposite to the equal side of the other triangle.

In the first of these two cases the student will easily see, after reading I. 29, that the two triangles are not necessarily equal. In the second case also the triangles are not necessarily equal, as will be seen from a proposition which we shall now demonstrate.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles opposite to one pair of equal sides equal, the angle opposite the other pair of equal sides shall be either equal or supplementary.*

Let  $ABC$ ,  $DEF$  be two triangles having  $AB = DE$ ,  $AC = DF$ , and the  $\angle ABC$  opposite  $AC =$  the  $\angle DEF$  opposite the equal side  $DF$ .

Let the  $\triangle ABC$  be applied to the  $\triangle DEF$  so that  $AB$  coincides with  $DE$  and the  $\angle ABC$  with the  $\angle DEF$ . Then the side  $BC$  falls on the side  $EF$ .



The point  $C$  then coincides with  $F$ , or falls in  $EF$ , or in  $EF$  produced.

If  $C$  coincides with  $F$ , the triangle  $ABC$  coincides with the triangle  $DEF$ , and they are equal in all respects.

If not, let  $C$  fall on the point  $G$  in  $EF$ , or  $EF$  produced.

The  $\triangle^s ABC, DEG$  then coincide and are equal in all respects, so that  $\angle DGE = \angle ACB$ , and the side  $DG =$  the side  $AC$ .

But  $AC = DF$ ;  $\therefore DG = DF$ ;  $\therefore \angle DGF = \angle DFG$ . [I. 6.]

But the  $\angle^s DFG, DFE =$  two right angles.

$\therefore$  the  $\angle^s DGE, DFE =$  two right angles;

$\therefore$  the  $\angle^s ACB, DFE =$  two right angles;

that is, the  $\angle^s ACB, DFE$  are supplementary.

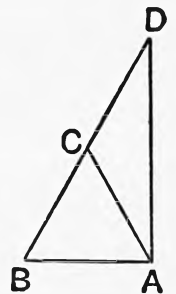
The student who wishes to examine the attempts that have been made to avoid the assumption made in Euclid's twelfth Axiom may consult Camerer's *Euclid*, Gergonne's *Annales de Mathématiques*, Volumes xv. and xvi., the work by Colonel Perronet Thompson entitled *Geometry without Axioms*, the article "Parallels" in the *English Cyclopædia*, a memoir by Professor Baden Powell in the second volume of the *Memoirs of the Ashmolean Society*, an article by M. Bouniakofsky in the *Bulletin de l'Académie Impériale*, Volume v., St. Petersburg, 1863, articles in the volumes of the *Philosophical Magazine* for 1856 and 1857, and a dissertation entitled *Sur un point de l'histoire de la Géométrie chez les Grecs*, par A. J. H. Vincent, Paris, 1857.

**I. 32.** If two triangles have two angles of the one equal to two angles of the other, each to each, they shall also have their third angles equal. This is a very important result, which is often required in the *Elements*. The student should notice how this result is established on Euclid's principles. By Axioms 11 and 2 one pair of right angles is equal to any other pair of right angles. Then, by I. 32, the three angles of one triangle are together equal to the three angles of any other triangle. Then, by Axiom 2, the sum of the two angles of one triangle is equal to the sum of the two equal angles of the other; and then, by Axiom 3, the third angles are equal.

After I. 32 we can draw a straight line at right angles to a given straight line from its extremity, without producing the given straight line.

Let  $AB$  be the given straight line. It is required to draw from  $A$  a straight line at right angles to  $AB$ .

On  $AB$  describe the equilateral triangle  $ABC$ . Produce  $BC$  to  $D$ , so that  $CD$  may be equal to  $CB$ . Join  $AD$ . Then  $AD$  shall be at right angles to  $AB$ . For the  $\angle CAD =$  the  $\angle CDA$ , and the  $\angle CAB =$  the  $\angle CBA$ , by I. 5.  $\therefore$  the  $\angle BAD =$  the two angles  $ABD, BDA$ , by Axiom 2. Therefore the  $\angle BAD$  is a right  $\angle$ , by I. 32.



**I. 35.** The equality of the parallelograms in I. 35 is an equality of area, and not an identity of figure. Legendre proposed to use the word

*equivalent* to express the equality of area, and to restrict the word *equal* to the case in which magnitudes admit of superposition and coincidence. This distinction, however, has not been generally adopted, probably because there are few cases in which any ambiguity can arise; in such cases we may say especially, *equal in area*, to prevent misconception.

Cresswell, in his *Treatise of Geometry*, has given a demonstration of I. 35, which shews that the parallelograms may be divided into pairs of pieces admitting of superposition and coincidence; see also his Preface, page x.

**I. 38.** An important case of I. 38 is that in which the triangles are on equal bases and have a *common* vertex.

**I. 40.** We may demonstrate I. 40 without adopting the indirect method. Join BD, CD. The triangles DBC and DEF are equal, by I. 38; the triangles ABC and DEF are equal, by hypothesis; therefore the triangles DBC and ABC are equal, by the first Axiom. Therefore AD is parallel to BC, by I. 39.

## THE SECOND BOOK.

The second book is devoted to the investigation of relations between the rectangles contained by straight lines divided into segments in various ways.

II. 2 and II. 3 are particular cases of II. 1.

**II. 11.** The student should notice that II. 11 gives a geometrical construction for the solution of a particular quadratic equation.

For if  $AB = a$ , and  $AH$ , the part to be found,  $= x$ , then  $HB = a - x$ , and the relation given, namely,  $\text{rect. } AB, BH = \text{sq. on } AH$ , says that

$$a(a - x) = x^2; \quad \therefore x^2 + ax = a^2,$$

that is, 
$$x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}.$$

Taking the square root, we have

$$x + \frac{a}{2} = \pm \frac{\sqrt{5}}{2}a, \text{ that is, } x = \frac{\sqrt{5}-1}{2}a \text{ or } -\frac{\sqrt{5}+1}{2}a.$$

The first value of  $x$  corresponds to the position of  $H$  found in the proposition. The second value corresponds to the position found in the first exercise following the proposition.

**II. 12, II. 13.** These are interesting in connection with I. 47; and, as the student may see hereafter, they are of great importance in Trigonometry; they are however not required in any of the parts of Euclid's *Elements* which are usually read. The converse of I. 47 is

proved in I. 48 ; and we can easily shew that converses of II. 12 and II. 13 are true.

Take the following, which is the converse of II. 12 : *if the square described on one side of a triangle be greater than the sum of the squares described on the other two sides, the angle opposite to the first side is obtuse.*

For the angle cannot be a right angle, since the square described on the first side would then be equal to the sum of the squares described on the other two sides, by I. 47 ; and the angle cannot be acute, since the square described on the first side would then be less than the sum of the squares described on the other two sides, by II. 13 ; therefore the angle must be obtuse.

Similarly, we may demonstrate the following, which is the converse of II. 13 : *if the square described on one side of a triangle be less than the sum of the squares described on the other two sides, the angle opposite to the first side is acute.*

II. 14. This is not required in any of the parts of Euclid's *Elements* which are usually read ; it is included in VI. 22.

### THE THIRD BOOK.

The third book of the *Elements* is devoted to properties of circles.

III. 1. In the construction, DC is said to be *produced* to E ; this assumes that D is within the circle, which Euclid demonstrates in III. 2.

III. 3. This consists of two parts, each of which is the converse of the other ; and the whole proposition is the converse of the corollary in III. 1.

The following proposition is analogous to III. 7 and III. 8.

*If any point be taken on the circumference of a circle, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is ; and of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote ; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the greatest line.*

The first two parts of this proposition are contained in III. 15 ; all three parts might be demonstrated in the manner of III. 7.

III. 9. Euclid has given three demonstrations of III. 9, of which Simson has chosen the second. Euclid's other demonstration is as follows. Join D with the middle point of the straight line AB ; then

it may be shewn that this straight line is at right angles to AB; and therefore the centre of the circle must lie in this straight line, by III. 1, Corollary. In the same manner it may be shewn that the centre of the circle must lie in the straight line which joins D with the middle point of the straight line BC. The centre of the circle must therefore be at D, because two straight lines cannot have more than one common point.

III. 10. Euclid has given two demonstrations of III. 10, of which Simson has chosen the second. Euclid's first demonstration resembles his first demonstration of III. 9. He shews that the centre of each circle is on the straight line which joins K with the middle point of the straight line BG, and also on the straight line which joins K with the middle point of the straight line BH; therefore K must be the centre of each circle.

The demonstration which Simson has chosen requires some additions to make it complete. For the point K might be supposed to fall *without* the circle DEF, or *on* its circumference, or *within* it; and of these three suppositions Euclid only considers the last. If the point K be supposed to fall *without* the circle DEF we obtain a contradiction of III. 8; which is absurd. If the point K be supposed to fall *on* the circumference of the circle DEF we obtain a contradiction of the proposition which we have enunciated at the end of the note on III. 7 and III. 8; which is absurd.

What is demonstrated in III. 10 is that the circumference of two circles cannot have more than *two* common points; there is nothing in the demonstration which assumes that the circles *cut* one another, but the enunciation refers to this case only because it is shewn in III. 13 that if two circles *touch* one another, their circumference cannot have more than *one* common point.

III. 11 may be deduced from III. 7. For GH is the least line that can be drawn from G to the circumference of the circle whose centre is F, by III. 7. Therefore GH is less than GA, that is, less than GD; which is absurd. Similarly, III. 12 may be deduced from III. 8.

III. 18. It does not appear that III. 18 adds anything to what we have already obtained in III. 16. For in III. 16 it is shewn that there is only one straight line which touches a given circle at a given point, and that the angle between this straight line and the radius drawn to the point of contact is a right angle.

III. 20. An important extension may be given to III. 20 by introducing angles greater than two right angles. For, in the first figure, suppose we draw the straight lines BF and CF. Then, the  $\angle BEA$  is double of the angle  $\angle BFA$ , and the  $\angle CEA$  is double of the  $\angle CFA$ ;

$\therefore$  the sum of the angles BEA and CEA is double of the  $\angle$ BFC. The sum of the angles BEA and CEA is greater than two right angles; we will call the sum the *re-entrant*  $\angle$ BEC. Thus the re-entrant  $\angle$ BEC is double of the  $\angle$ BFC. (See note on I. 32.) If this extension be used some of the demonstrations in the third book may be abbreviated. Thus III. 21 may be demonstrated without making two cases; III. 22 will follow immediately from the fact that the sum of the angles at the centre is equal to four right angles; and III. 31 will follow immediately from III. 20.

III. 21. In III. 21 Euclid himself has given only the first case; the second case has been added by Simson and others. In either of the figures of III. 21 if a point be taken on the same side of BD as A, the angle contained by the straight lines which join this point to the extremities of BD is *greater* or *less* than the angle BAD, according as the point is *within* or *without* the circle BAD; this follows from I. 21.

III. 32. The converse of III. 32 is true and important; namely, *if a straight line meet a circle, and from the point of meeting a straight line be drawn cutting the circle, and the angle between the two straight lines be equal to the angle in the alternate segment of the circle, the straight line which meets the circle shall touch the circle.*

This may be demonstrated indirectly. For, if possible, suppose that the straight line which meets the circle does not touch it. Draw through the point of meeting a straight line to touch the circle. Then, by III. 32 and the hypothesis, it will follow that two different straight lines pass through the same point, and make the same angle, on the same side, with a third straight line which also passes through that point; but this is impossible.

III. 35, III. 36. The following proposition constitutes a large part of the demonstrations of III. 35 and III. 36. *If any point be taken in the base, or the base produced, of an isosceles triangle, the rectangle contained by the segments of the base is equal to the difference of the square on the straight line joining this point to the vertex and the square on the side of the triangle.*

This proposition is in fact demonstrated by Euclid, without using any property of the circle; if it were enunciated and demonstrated before III. 35 and III. 36 the demonstrations of these two propositions might be shortened and simplified.

## THE FOURTH BOOK.

The fourth Book of the *Elements* consists entirely of problems. The first five propositions relate to triangles of any kind; the remaining



propositions relate to polygons which have all their sides equal and all their angles equal.

IV. 3. We can also describe a triangle equiangular to a given triangle, and such that one of its sides and the other two sides produced shall touch a given circle. For, in the figure of IV. 3, suppose AK produced to meet the circle again; and at the point of intersection draw a straight line touching the circle; this straight line, with parts of NB and NC, will form a triangle, which will be equiangular to the triangle MLN, and therefore equiangular to the triangle EDF; and one of the sides of this triangle, and the other two sides produced, will touch the given circle.

It was first demonstrated by Gauss in 1801, in his *Disquisitiones Arithmeticae*, that it is possible to describe geometrically a regular polygon of  $2^n + 1$  sides, provided  $2^n + 1$  be a prime number; the demonstration is not of an elementary character. As an example, it follows that a regular polygon of seventeen sides can be described geometrically; this example is discussed in Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*.

### THE FIFTH BOOK.

The fifth Book of the *Elements* is on *Proportion*. Much has been written respecting Euclid's treatment of this subject; besides the Commentaries on the *Elements* to which we have already referred, the student may consult the articles "Ratio" and "Proportion" in the *English Cyclopædia*, and the tract on the *Connexion of Number and Magnitude* by Professor De Morgan.

The fifth Book relates not merely to length and space, but to any kind of magnitude of which we can form multiples.

V. Def. 3. Simson considers that the definitions 3 and 8 are "not Euclid's, but added by some unskilful editor." Other commentators also have rejected these definitions as useless. The last word of the third definition should be *quantuplicity*, not *quantity*; so that the definition indicates that ratio refers to the *number of times* which one magnitude contains another. See De Morgan's *Differential and Integral Calculus*, page 18.

*Compound Ratio.* The definition of compound ratio was supplied by Simson. The Greek text does not give any definition of compound ratio here, but gives one as the fifth definition of the sixth Book, which Simson rejects as absurd and useless.

V. Defs. 18, 19, 20. The definitions 18, 19, 20 are not presented by Simson precisely as they stand in the original. The last sentence in

definition 18 was supplied by Simson. Euclid does not connect definitions 19 and 20 with definition 18. In 19 he defines *ordinate proportion*, and in 20 he defines *perturbate proportion*. Nothing would be lost if Euclid's definition 18 were entirely omitted, and the term *ex æquali* never employed. Euclid employs such a term in the enunciations of V. 20, 21, 22, 23 ; but it seems quite useless, and is accordingly neglected by Simson and others in their translations.

### THE SIXTH BOOK.

The sixth Book of the *Elements* consists of the application of the theory of proportion to establish properties of geometrical figures.

VI. Def. 3. The third definition is useless, for Euclid makes no mention of reciprocal figures.

VI. 2. The enunciation of this important proposition is open to objection, for the manner in which the sides may be cut is not sufficiently limited. Suppose, for example, that AD is double of DB, and CE double of EA ; the sides are then cut proportionally, for each side is divided into two parts, one of which is double of the other ; but DE is not parallel to BC. It should therefore be stated in the enunciation that *the segments terminated at the vertex of the triangle are to be homologous terms in the ratios, that is, are to be the antecedents or the consequents of the ratios.*

VI. A. This proposition was supplied by Simson.

VI. 4. By superposition we might deduce VI. 4 immediately from VI. 2.

VI. 10. The most important case of this proposition is that in which a straight line is to be divided either *internally* or *externally* into two parts which shall be in a given ratio.

The case in which the straight line is to be divided *internally* is given in the text ; suppose, for example, that the given ratio is that of AE to EC ; then AB is divided at G in the given ratio.

Suppose, however, that AB is to be divided *externally* in a given ratio ; that is, suppose that AB is to be produced so that the whole straight line made up of AB and the part produced may be to the part produced in a given ratio. Let the given ratio be that of AC to CE. Join EB ; through C draw a straight line parallel to EB ; then this straight line will meet AB, produced through B, at the required point.

VI. 11. This is a particular case of VI. 12.



**VI. 14.** The following is a full exhibition of the steps which lead to the result that FB and BG are in one straight line.

The  $\angle DBF = \text{the } \angle GBE$ ; add to each the  $\angle FBE$ ;

$\therefore$  the  $\angle^s GBE, FBE = \text{the } \angle^s DBF, FBE = \text{two rt. } \angle^s$ .

$\therefore$  FB and BG are in one straight line.

[I. 14.]

**VI. 15.** This may be inferred from VI. 14, since a triangle is half of a parallelogram with the same base and altitude.

It is not difficult to establish a third proposition conversely connected with the two involved in VI. 14, and a third proposition similarly conversely connected with the two involved in VI. 15. These propositions are the following.

*Equal parallelograms which have their sides reciprocally proportional, have their angles equal, each to each.*

*Equal triangles which have the sides about a pair of angles reciprocally proportional, have also angles equal or together equal to two right angles.*

We will take the latter proposition.

Let ABC, ADE be equal triangles; and let CA be to AD as AE is to AB: either the angle BAC shall be equal to the angle DAE, or the angles BAC and DAE shall be together equal to two right angles.

[The student can construct the figure for himself.]

Place the triangles so that CA and AD may be in one straight line; then if EA and AB are in one straight line the  $\angle BAC = \angle DAE$ , [I. 15. If EA and AB are not in one straight line, produce BA through A to F, so that AF may = AE; join DF and EF.

Then because CA is to AD as AE to AB,  
and AF = AE,

[Hypothesis.

[Construction.

$\therefore$  CA is to AD as AF to AB.

$\therefore$  the  $\triangle DAF = \text{the } \triangle BAC = \triangle DAE$ .

[VI. 15 and Ax. 1.

$\therefore$  EF is parallel to AD.

[I. 39.]

Suppose now that the  $\angle DAE > \text{the } \angle DAF$ .

Then the  $\angle CAE = \text{the } \angle AEF = \angle AFE = \angle BAC$ .

[I. 29.]

$\therefore$  the angles BAC and DAE together = two right angles. [I. 29 and 5.]

Similarly, if the  $\angle DAE$  is less than the  $\angle DAF$ .

**VI. 16.** This is a particular case of VI. 14.

**VI. 17.** This is a particular case of VI. 16.

**VI. 22.** There is a step in the second part of VI. 22 which requires examination. After it has been shewn that the figure SR is equal to the similar and similarly situated figure NH, it is added "therefore PR is equal to GH." In the Greek text reference is here made to a *lemma* which follows the proposition. The word *lemma* is occasionally used in mathematics to denote an auxiliary proposition. From the unusual

circumstance of a reference to something following, Simson probably concluded that the lemma could not be Euclid's, and accordingly he takes no notice of it. The following is the substance of the lemma.

If PR be not equal to GH, one of them must be greater than the other; suppose PR greater than GH.

Then, because SR and NH are similar figures, PR is to PS as GH is to GN. [VI. *Definition 1*.

But  $PR > GH$ ;  $\therefore PS > GN$ . [Hypothesis.

$\therefore$  the  $\triangle RPS >$  the  $\triangle HGN$ . [I. 4, *Axiom 9*.

But, because SR and NH are similar, the  $\triangle RPS =$  the  $\triangle HGN$ ; [VI. 20. which is impossible.  $\therefore PR = GH$ .

**VI. 23.** In the figure of VI. 23 suppose BD and GE drawn. Then the  $\triangle BCD$  is to the  $\triangle GCE$  as AC to the  $\parallel^m$  CF. Hence the result may be extended to triangles, and we have the following theorem; *Triangles which have one angle of the one equal to one angle of the other have to one another the ratio which is compounded of the ratios of their sides.*

Then VI. 19 is an immediate consequence of this theorem. For let ABC and DEF be similar triangles, so that AB is to BC as DE to EF; and therefore, alternately, AB is to DE as BC to EF. Then by the theorem, the  $\triangle ABC$  has to the  $\triangle DEF$  the ratio which is compounded of the ratios of AB to DE and of BC to EF, that is, the ratio which is compounded of the ratios of BC to EF and of BC to EF. And, from the definitions of duplicate ratio and of compound ratio (V. Def. 11), it follows that the ratio compounded of the ratios of BC to EF and of BC to EF is the duplicate ratio of BC to EF.

We have omitted in the sixth Book Propositions 27, 28, 29, and the first solution which Euclid gives of Proposition 30, as they appear now to be never required, and have been condemned as useless by various modern commentators. Some idea of the nature of these propositions may be obtained from the following statement of the problem proposed by Euclid in VI. 29. AB is a given straight line; it has to be produced through B to a point O, and a parallelogram described on AO subject to the following conditions; the parallelogram is to be equal to a given rectilineal figure, and the parallelogram on the base BO which can be cut off by a straight line through B is to be similar to a given parallelogram.

**VI. 32.** This proposition seems of no use. Moreover the enunciation is imperfect. For suppose ED to be produced through D to a point F, such that DF is equal to DE; and join CF. Then the triangle CDF will satisfy all the conditions in Euclid's enunciation, as well as the triangle CDE; but CF and CB are not in one straight line. It





**2.** *Two equal triangles are on the same base and on opposite sides of it; the straight line joining their vertices is bisected by the base, or the base produced.*

Let  $ABC, DBC$  be the equal triangles, and let  $AD$  meet  $BC$  in  $G$ .

Then shall  $AG = GD$ .

Draw  $AE, DF \perp$  to  $BC$  and produce  $DF$  to  $D'$  making  $FD' = DF$ . Then from the  $\triangle^s DCF, D'CF$  it is easily seen that  $D'C = CD$ ; so  $D'B = BD$ .

$\therefore$  the  $\triangle^s DBC, D'BC$  are equal in all respects.

$\therefore$  area  $\triangle D'BC = \text{area } \triangle DBC = \text{area } \triangle ABC$ .

[Hypothesis.]

$\therefore AD', BC$  are parallel (I. 39), and  $\therefore AEFD'$  is a  $\parallel^{\text{gm}}$ .  $\therefore AE = D'F = FD$ . [I. 34.]

$\therefore$  in the  $\triangle^s AGE, DGF$  we have  $\angle AGE = \angle DGF$ , rt.  $\angle AEG = \text{rt. } \angle DFG$ , and side  $AE = FD$ .  $\therefore AG = GD$ . [Q.E.D.]

The same proof holds if  $AD$  cuts  $BC$  produced.

**Corollary.** It follows that all triangles with equal areas and equal bases have equal heights.

**3.** *Any medium of a triangle bisects all lines which are parallel to the base to which it is drawn and which are intercepted by the sides.*

Let  $PQ, \parallel$  to  $BC$ , meet the median  $AD$  through  $A$  in  $R$ .

Then shall  $PR = RQ$ .

Since  $BD = DC$ ,  $\therefore$ , by I. 38,  $\triangle ABD = \triangle ADC$ , and  $\triangle PBD = \triangle QDC$ .

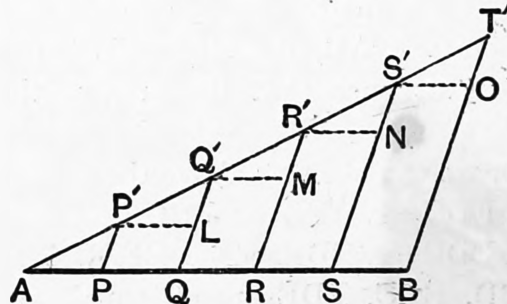
$\therefore$ , by subtraction,  $\triangle APD = \triangle AQD$ .

i.e. the  $\triangle^s APD, AQD$  on the same base  $AD$  are equal.

$\therefore$ , by the previous theorem,  $PQ$  is bisected by  $AD$ .

Also  $PQ$  is any straight line  $\parallel$  to the base.  $\therefore$  etc. [Q.E.D.]

**4.** *To divide a given straight line into any number of given parts.*



Let  $AB$  be the given straight line, and let it be required to divide it into five given parts.

Through  $A$  draw a straight line  $AT'$ , making any angle with  $AB$ .

On it take any point  $P'$  and cut off portions  $P'Q'$ ,  $Q'R'$ ,  $R'S'$ ,  $S'T'$  each equal to  $AP'$ .

Join  $T'B$ , and draw  $P'P$ ,  $Q'Q$ ,  $R'R$ ,  $S'S$  all  $\parallel$  to  $T'B$  to meet  $AB$  in  $P$ ,  $Q$ ,  $R$ ,  $S$  respectively.

Then  $AB$  shall be divided at  $P$ ,  $Q$ ,  $R$ ,  $S$  as required.

Draw  $P'L$ ,  $Q'M$ ,  $R'N$ ,  $S'O$   $\parallel$  to  $AB$  to meet  $Q'Q$ ,  $R'R$ ,  $S'S$ , and  $T'B$  in  $L$ ,  $M$ ,  $N$ ,  $O$ .

Then  $\angle Q'LP' = \angle LQP = \angle P'PA$ ,

and  $\angle Q'P'L = \angle P'AP$ .

[I. 29.]

$\therefore$  since  $AP' = P'Q'$ , the  $\triangle^s APP'$ ,  $P'LQ$  are equal in all respects.

Similarly with the  $\triangle^s Q'MR'$ ,  $R'NS'$ ,  $S'OT'$ .

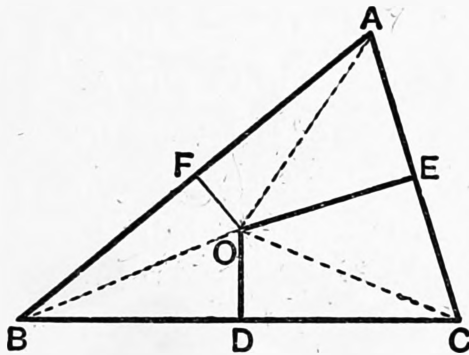
$\therefore AP = P'L = Q'M = R'N = S'O$ .

$\therefore AP = PQ = QR = RS = SB$ .

[I. 34.]

The method is the same whatever be the number of parts into which  $AB$  is to be divided.

✓ 5. *The straight lines drawn at right angles to the sides of a triangle from the points of bisection of the sides meet in a point.*



Let  $ABC$  be a  $\triangle$ ; bisect  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ ,  $F$  respectively. Draw  $DO$   $\perp^r$  to  $BC$  and  $EO$   $\perp^r$  to  $CA$ , and let them meet in  $O$ . Join  $OA$ ,  $OB$ ,  $OC$ ,  $OF$ .

In the  $\triangle^s BDO$ ,  $CDO$  we have  $BD = CD$ ,  $DO$  common, and the  $\text{rt. } \angle BDO = \text{the rt. } \angle CDO$ ;

[I. 4.]

$\therefore BO = CO$ .

Similarly, from the  $\triangle^s CEO$ ,  $AEO$  we have  $CO = AO$ ;

$\therefore AO = BO$ .

Thus the  $\triangle^s AFO$ ,  $BFO$  have their sides respectively equal;

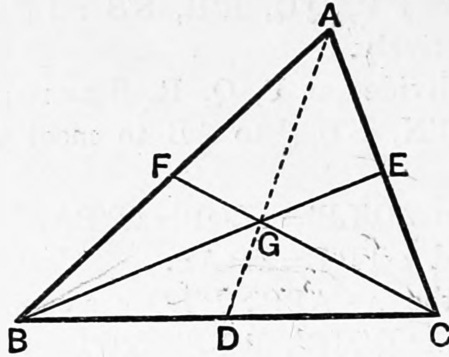
$\therefore$  the  $\angle^s AFO$ ,  $BFO$  are equal, and  $\therefore$  each is a  $\text{rt. } \angle$ ; [I. 8 and *Def. 10*.

$\therefore FO$  bisects  $AB$  at  $\text{rt. } \angle^s$ ;  $\therefore$  etc.

**Corollary.** Since  $AO = BO = CO$ , the point  $O$  is equidistant from the angular points of the  $\triangle$ .

When three straight lines, such as  $OA$ ,  $OB$ ,  $OC$  (or  $DO$ ,  $EO$ ,  $FO$ ), meet in a point they are said to be **concurrent**.

6. *The straight lines drawn from the angles of a  $\triangle$  to bisect the opposite sides meet in a point.*



Let  $ABC$  be a  $\triangle$ ;  $D, E, F$  the middle points of its sides; let  $BE, CF$  meet in  $G$ .

Join  $AG, GD$ ; *they shall be in the same straight line.*

The  $\triangle BEA = \triangle BEC$ , and  $\triangle GEA = \triangle GEC$ ; [I. 38.]

$\therefore$  by subtraction,  $\triangle BGA = \triangle BGC$ .

Similarly,  $\triangle CGA = \triangle BGC$ ;

$\therefore \triangle BGA = \triangle CGA$ .

Also  $\triangle BGD = \triangle CGD$ ; [I. 38.]

$\therefore$  the  $\triangle^s BGA, BGD$  together  $= \triangle^s CGA, CGD$ ;

$\therefore$  the  $\triangle^s BGA, BGD$  together  $=$  half the  $\triangle ABC$   
 $= \triangle BDA$ . [I. 38.]

$\therefore G$  must fall on the straight line  $AD$ , otherwise the whole would be equal to its part;

that is,  $AG, GD$  are in one straight line. [I. 38.]

7. *To prove  $AG = 2 GD, BG = 2 GE$ , and  $CG = 2 GF$ .*

By the previous article,  $\triangle AGB = \triangle BGC$ ,  
and  $\triangle BGC =$  sum of  $\triangle^s BGD, CGD =$  twice  $\triangle BGD$ . [I. 38.]

$\therefore \triangle AGB =$  twice  $\triangle BGD$ .

If  $K$  be the middle point of  $AG$ , we then have, by I. 38,

$\triangle ABK = \triangle KBG = \triangle GBD$ ;

$\therefore AK = KG = GD$ ;  $\therefore AG = 2 GD$ .

Similarly  $BG = 2 GE$ , and  $CG = 2 GF$ .

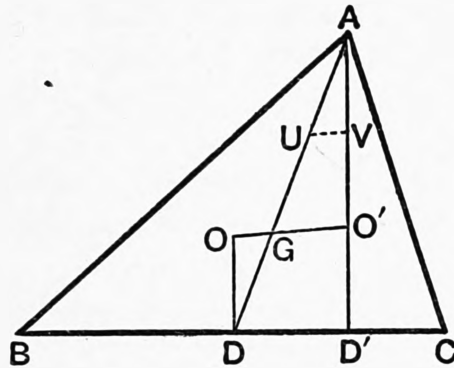
This point  $G$  is called the **Centroid** of the  $\triangle ABC$ , and  $AD, BE, CF$  are the **Medians**.

**8.** *The perpendiculars drawn from the vertices of a triangle upon the opposite sides meet in a point.*

Let  $ABC$  be a  $\triangle$ ,  $O$  the point in which meet the perp<sup>rs</sup> to the sides through the middle points,  $G$  the point in which meet the straight lines joining the vertices to opposite sides. [Arts. 5, 6.]

Let  $D$  be the middle point of  $BC$  and  $AD'$  the perp<sup>r</sup> from  $A$  on  $BC$ .

Let  $OG$  meet  $AD'$  in  $O'$ , and let  $U$ ,  $V$  be the middle points of  $AG$ ,  $AO'$ .



Then  $UV$  is parallel to, and one-half of,  $GO'$ . [Art. 1.]

Because  $AD'$ ,  $OD$  are parallel, both being perp<sup>r</sup> to  $BC$ , [I. 28.]

$\therefore \angle ADO = \angle UAV$ . [I. 29.]

Also, since  $UV$ ,  $GO'$  are parallel,  $\therefore \angle AUV = \angle AGO' = \angle OGD$ .

Also,  $AU = \frac{1}{2} AG = GD$ ; [Art. 7.]

$\therefore$  the  $\triangle^s$   $AUV$ ,  $DGO$  are equal in all respects; [I. 4.]

$\therefore OG = UV = \frac{1}{2} GO'$ ;  $\therefore GO' = 2 OG$ .

The point in which  $OG$  meets the perp<sup>r</sup> from  $A$  on  $BC$  is thus a fixed point on  $OG$  produced; similarly the perp<sup>rs</sup> from  $B$  on  $CA$ , and from  $C$  on  $AB$  pass through this same fixed point  $O'$ ;

Hence the three perpendiculars from  $A$ ,  $B$ , and  $C$ , upon the opposite sides meet in a point  $O'$  which lies on  $OG$  produced, and is such that  $GO' = 2 OG$ .

This point  $O'$  is called the **Orthocentre**.

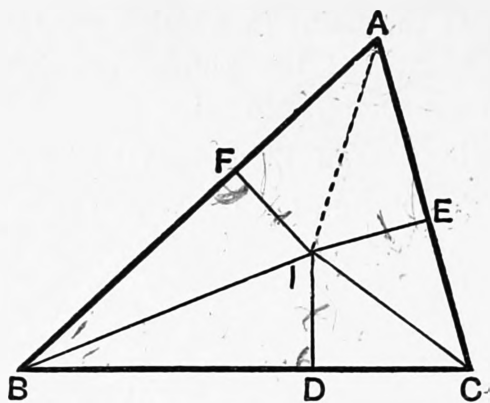
**Corollary.** Since the triangles  $AUV$ ,  $DGO$  are equal,

$\therefore OD = AV = \frac{1}{2} AO'$ ;

$\therefore AO' = 2 OD$ .



✓ 9. *The straight lines which bisect the angles of a triangle meet in a point.*



Let  $ABC$  be a triangle; bisect the angles at  $B$  and  $C$  by straight lines meeting at  $I$ ; join  $AI$ :

*then  $AI$  shall bisect the angle at  $A$ .*

From  $I$  draw  $ID$  perpendicular to  $BC$ ,  $IE$  perpendicular to  $CA$ , and  $IF$  perpendicular to  $AB$ .

In the  $\triangle^s BIF, BID$  we have  $BI$  common,  $\angle DBI = \angle FBI$ , and the rt.  $\angle BDI = \text{rt. } \angle BFI$ ;

$$\therefore IF = ID.$$

[Construction.  
[I. 26.

Similarly from the  $\triangle^s CID, CIE$  we have  $ID = IE$ ;

$$\therefore IE = IF.$$

Then since  $E, F$  are right  $\angle^s$ , we have  $AF^2 + FI^2 = AI^2 = AE^2 + IE^2$ . [I. 47.  
 $\therefore AF = AE$ , and the  $\triangle^s AEI, AFI$  are equal in all respects;

$$\therefore \angle IAE = \angle IAF.$$

**Corollary.** Since  $ID = IE = IF$ , the point  $I$  is such that its perpendicular distances from the three sides of the  $\triangle$  are equal.

10. *Let two sides of a triangle be produced through the base; then the straight lines which bisect the two exterior angles thus formed, and the straight line which bisects the vertical angle of the triangle, meet at the same point.*

This may be shewn in the same manner as the last theorem.

11. *If two medians of a triangle be equal, the angles through which they are drawn shall be equal.*

In the figure of Art. 6, let  $BE = CF$ . Then, by Art. 7,  $GB = GC$ . Hence, since  $BD = DC$ , the  $\triangle^s GDB, GDC$  have their sides respectively equal.

Hence they are equal in all respects,

and  $\therefore \angle GDB = \angle GDC$ .

[I. 8.]

$\therefore$  GDB, GDC are both right  $\angle^s$ .

[Def. 10.]

Hence from the  $\triangle^s$  ADB, ADC we have

$AB = AC$ .

[Ax. 11, and I. 4.]

$\therefore \angle ABC = \angle ACB$ .

[I. 5.]

## EXERCISES.

1. The sides AB, AC of a triangle ABC are bisected at D, E; BE and CD are drawn and produced to F and G so that  $EF = BE$  and  $DG = DC$ ; prove that AF and AG are in the same straight line.

[Use Art. 1.]

2. The middle points of the sides BC, CA, and AB of a triangle are D, E, F; FG is drawn parallel to BE to meet DE in G. Prove that the sides of the triangle CFG are equal to the medians of the triangle ABC, and hence that the sum of any two medians of a triangle is always greater than the third.

3. Prove that the sum of the medians of a triangle is less than the perimeter, but greater than three quarters of the perimeter, of the triangle.

$[GB + GC > BC, \text{ i.e. } \frac{2}{3}BE + \frac{2}{3}CF > BC, \text{ etc.}]$

Also  $AD < AE + ED, \text{ i.e. } < \frac{1}{2}AC + \frac{1}{2}AB, \text{ etc. (Fig. Art. 6).]$

4. An angle of a triangle is right, acute, or obtuse according as the median drawn through it is  $=$ ,  $>$ , or  $<$  half the side it bisects.

5. If one side of a triangle be longer than another, the corresponding median is shorter.

[Apply I. 25 to the  $\triangle^s$  ADB, ADC (Fig. Art. 6), and then I. 24 to the  $\triangle^s$  GDB, GDC.]

6. Of the two angles formed by a median with the two adjacent sides, that made with the shorter side is the greater.

7. If G be the centroid of a triangle ABC, the triangle BGC is equal in area to the quadrilateral AEGF. [Fig. Art. 6.]

## ON GEOMETRICAL ANALYSIS.

**12.** The substantives *analysis* and *synthesis*, and the corresponding adjectives *analytical* and *synthetical*, are of frequent occurrence in mathematics. In general *analysis* means decomposition, or the separating a whole into its parts, and *synthesis* means composition, or making a whole out of its parts. In Geometry, however, these words are used in a more special sense. In **synthesis** we begin with results already established, and end with some new result; thus, by the aid of theorems already demonstrated, and problems already solved, we demonstrate some new theorem, or solve some new problem. In **analysis** we begin with assuming the truth of some theorem or the solution of some problem; and we deduce from the assumption consequences which we can compare with results already established, and thus test the validity of our assumption.

**13.** The propositions in Euclid's Elements are all exhibited synthetically; the student is only employed in examining the soundness of the reasoning by which each successive addition is made to the collection of geometrical truths already obtained; and there is no hint given as to the manner in which the propositions were originally discovered. Some of the constructions and demonstrations appear rather artificial, and we are thus naturally induced to enquire whether any rules can be discovered by which we may be guided easily and naturally to the investigation of new propositions.

**14.** Geometrical analysis has sometimes been described in language which might lead to the expectation that directions could be given which would enable a student to proceed to the demonstration of any proposed theorem, or the solution of any proposed problem, with confidence of success; but no such directions can be given. We will state the exact extent of these directions. Suppose that a new theorem is proposed for investigation, or a new problem for trial. Assume the truth of the theorem or the solution of the problem, and deduce consequences from this assumption combined with results which have been already established. If a consequence can be deduced which contradicts some result already established, this amounts to a demonstration that our assumption is inadmissible; that is, the theorem is not true, or the problem cannot be solved. If a consequence can be deduced which coincides with some result already established, we cannot say that the assumption is inadmissible; and it *may happen* that by

starting from the consequence which we deduced, and retracing our steps, we can succeed in giving a synthetical demonstration of the theorem, or solution of the problem. These directions, however, are very vague, because no certain rule can be prescribed by which we are to combine our assumption with results already established ; and moreover no test exists by which we can ascertain whether a valid consequence which we have drawn from an assumption will enable us to establish the assumption itself. That a proposition may be false and yet furnish consequences which are true, can be seen from a simple example. Suppose a theorem were proposed for investigation in the following words ; *one angle of a triangle is to another as the side opposite to the first angle is to the side opposite to the other*. If this be assumed to be true we can immediately deduce Euclid's result in I. 19 ; but from Euclid's result in I. 19 we cannot retrace our steps and establish the proposed theorem, and in fact the proposed theorem is false.

Thus the only definite statement in the directions respecting geometrical analysis is, that if a consequence can be deduced from an assumed proposition which contradicts a result already established, that assumed proposition must be false.

**15.** We may mention, in particular, that a consequence would contradict results already established, if we could shew that it would lead to the solution of a problem already given up as impossible. There are three famous problems which are now admitted to be beyond the power of Geometry ; namely :

To find a straight line equal in length to the circumference of a given circle,

To trisect any given angle, and

To find two mean proportionals between two given straight lines.

The grounds on which the geometrical solution of these problems is admitted to be impossible cannot be explained without a knowledge of the higher parts of mathematics ; the student may, however, be content with the fact that innumerable attempts have been made to obtain solutions, and that these attempts have been made in vain.

The first of these problems is usually referred to as the *Quadrature of the Circle*. For the history of it the student should consult the article in the *English Cyclopædia* under that head, and also a series of papers in the *Athenæum* for 1863 and subsequent years, entitled a *Budget of Paradoxes*, by Professor De Morgan.

The third of the three problems is often referred to as the *Duplication of the Cube*.

We will now give some examples of geometrical analysis.

✓ **16.** *From two given points it is required to draw to the same point in a given straight line two straight lines equally inclined to the given straight line.*

Let A and B be the given points, and CD the given straight line.

Suppose AE and EB to be the two straight lines equally inclined to CD. Draw BF perpendicular to CD, and produce AE and BF to meet at G.

Then the  $\angle BED = \text{the } \angle AEC$ , by hypothesis; and

the  $\angle AEC = \text{the } \angle DEG$ . [I. 15.]

Hence the  $\triangle^s$  BEF and GEF are equal in all respects; [I. 26.]

$\therefore FG = FB$ .

This result gives the required

**Construction.** Draw BF perpendicular to CD, and produce it to G, so that FG may be equal to FB; then join AG, and AG will intersect CD at the required point.

For the  $\triangle^s$  BEF, GEF are equal in all respects, and

$\therefore \angle BEF = \angle GEF = \angle AEC$ .

If A be on the opposite side of CD from B, join GA, and let it be produced to meet CD in K. Then AK, BK are equally inclined to CD.

✓ **17.** *To divide a given straight line into two parts such that the difference of the squares on the parts may be equal to a given square.*

Let AB be the given straight line, and suppose C the required point.

Then  $AC^2 - CB^2 = \text{given sq.}$ ;  $\therefore AC^2 = CB^2 + \text{given square}$ .

Erect at B a perpendicular BD, so that

$BD^2 = \text{given sq.}$

$\therefore AC^2 = CB^2 + BD^2 = CD^2$ ; [I. 47.]

$\therefore AC = CD$ , and the  $\triangle ACD$  is isosceles.

The  $\angle CAD$  therefore = the  $\angle ADC$ .

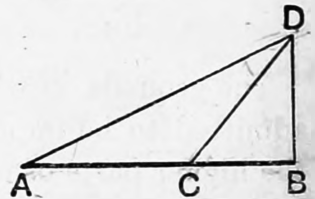
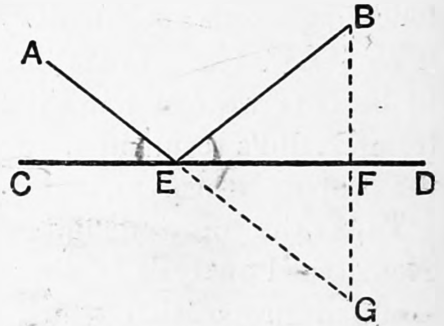
Hence we have the following

**Construction.** At B draw  $BD \perp$  to AB and equal to the side of the given sq. Join AD, and make the  $\angle ADC$  equal to the  $\angle BAD$ , and let DC meet AB in C. Then C is the required point.

For  $AC^2 = CD^2 = CB^2 + BD^2$ .

[I. 47.]

It is obvious that the given square must not exceed the square on AB, in order that the problem may be possible.



There are two positions of C, if it is not specified which of the two segments AC and CB is to be greater than the other; but only one position, if it is specified.

In like manner we may solve the problem, to produce a given straight line so that the square on the whole straight line made up of the given straight line and the part produced, may exceed the square on the part produced by a given square, which is not less than the square on the given straight line. For, in this case,  $BD > BA$ ;  $\therefore \angle BDA < \angle BAD$ , and DC therefore meets AB produced.

The two problems may be combined in one enunciation thus, to divide a given straight line internally or externally so that the difference of the squares on the segments may be equal to a given square.

✓ **18.** Given two intersecting straight lines, OA and OB, and any point P between them; to draw through P a straight line such that the portion of it intersected between OA and OB is bisected at P.

Let QPR be the required line. Join OP and produce to S so that  $OP = PS$ . Then QR and OS bisect one another at P, and this is a property of the diagonals of a  $\parallel^m$ . Hence

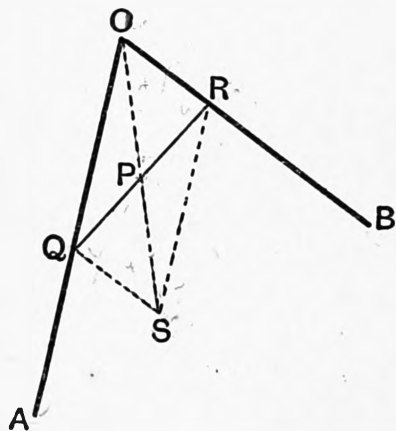
**Construction.** Join OP and produce it to S so that  $OP = PS$ . Draw  $\parallel^s$  SQ, SR as in the figure. Then SQOR is a  $\parallel^m$ ;

$\therefore$  SO, QR bisect one another.

But P is the middle point of OS.

$\therefore$  P is the middle point of QR.

$\therefore$  QPR is drawn as required.



✓ **19.** Bisect a triangle ABC by a straight line drawn through a given point P in the base BC.

Let PE be the required straight line.

Then area ABPE = half the  $\triangle ABC$ .

But, if D be the middle point of BC,

$\triangle ADB =$  half the  $\triangle ABC$ . [I. 38.]

$\therefore ABPE = \triangle ADB$ .

$\therefore$  subtracting  $\triangle ABP$  from each, we have

$\triangle AEP = \triangle ADP$ .

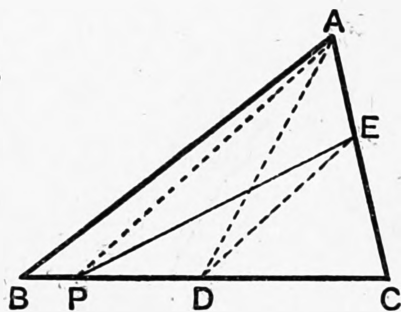
$\therefore AP, DE$  are parallel. [I. 39.]

Hence the construction required: Join AP; through D, the middle point of BC, draw  $DE \parallel$  to PA to meet AC in E; then PE is the required straight line.

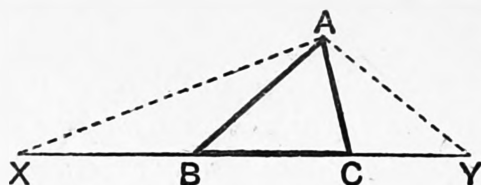
For  $AEPB = \triangle AEP + \triangle ABP = \triangle ADP + \triangle ABP$

$= \triangle ADB =$  half of  $\triangle ABC$ .

[I. 37.]



**20.** Construct a triangle, given its perimeter and its base angles.



Suppose that  $ABC$  is the required  $\triangle$ ; produce  $CB$  to  $X$  and  $BC$  to  $Y$ , so that  $BX = BA$  and  $CY = CA$ . Then  $XY$  is equal to the given perimeter, and  $\angle BXA = \angle BAX$ , and  $\therefore \angle ABC = \text{twice } \angle AXY$ . 132

$\therefore \angle AXY = \text{half } \angle ABC$ , and similarly  $\angle AYX = \text{half } \angle ACB$ .

**Hence the construction required**; Let  $XY$  equal the given perimeter; at  $X$  make  $\angle YXA = \text{half one given base } \angle$ , and at  $Y$  make  $\angle XYA = \text{half the other given base } \angle$ . At  $A$  make  $\angle XAB = \angle AXB$  and  $\angle YAC = \angle AYC$  where  $AB, AC$  lie between  $AX, AY$  and meet  $XY$  in  $B, C$ . Clearly  $ABC$  is now the required  $\triangle$ .

## EXERCISES.

1. Construct an isosceles triangle having given the base and the perpendicular on it from the opposite angle.

Construct a right-angled triangle having given

2. the hypotenuse and an acute angle;
3. the hypotenuse and a side;
4. the hypotenuse and the sum of the remaining sides;
5. the hypotenuse and the difference of the other sides;
6. the hypotenuse and the perp<sup>r</sup> on it from the right angle. [Use Page 60, Ex. 8.]
7. a side and the perpendicular on the hypotenuse from the right angle;
8. the perimeter and an acute angle. [Use Art. 20.]
9. Construct a triangle having given the middle points of its sides.
10. Construct a triangle, given its three medians.

[Let  $ABC$  be a  $\triangle$ ,  $G$  its centroid and  $AD$  the median through  $A$ ; complete the  $\parallel^{\text{gm}} BGCO$ ; then  $GO$  is bisected at  $D$  and  $GCO$  is a  $\triangle$  having its sides respectively equal to two-thirds of the medians. Hence the construction.]

11. Construct a triangle, given the base, the difference of the angles at the base, and the difference of its sides.

[Let  $BC$  be the given base; make  $\angle BCD = \frac{1}{2}$  the given diff. of the base  $\angle^s$ , and take  $D$  on  $CD$  such that  $BD = \text{the given difference of the sides}$ ; produce  $BD$  to  $A$ , making  $\angle ACD = \angle ADC$ . Then  $ABC$  is the required  $\triangle$ .]

**12.** Construct a triangle, given its base, one of its base angles, and (1) the sum, (2) the difference of its sides.

**13.** Bisect a parallelogram by a straight line drawn through a given point in its plane.

**14.** Trisect a right angle.

**15.** Bisect a quadrilateral by a straight line drawn through (1) an angular point, (2) a point in one of the sides.

[(1) Let ABCD be the quad<sup>l</sup>, B being nearer to AC than D; through E, the middle point of BD draw a line  $\parallel$  to AC to meet DC in G; then AG bisects the quad<sup>l</sup>. (2) Let the given point be P in AB; draw DE  $\parallel$  to AC to meet BA in E; through F, the mid-point of BE, draw FQ  $\parallel$  to CP to meet CD in Q; then FQ bisects the quad<sup>l</sup>.]

**16.** Construct a parallelogram, given one side and the two diagonals.

**17.** Trisect a given straight line. [On the given straight line AB describe an equilateral  $\triangle$  ABC, and through its centroid G draw GX, GY  $\parallel$  to AC, CB to meet AB in X, Y.]

**18.** Trisect a triangle by a straight line through an angular point.

[Use Ex. 17.]

**19.** Trisect a triangle by a straight line drawn through a point in one side. [Let P be the point in the side BC, and X, Y the points of trisection of BC (Ex 17); draw XR, YS  $\parallel$  to AP to meet the sides in R and S; then, as in Art. 19, PR and PS divide the  $\triangle$  as required.]

**20.** On the side AB of a parallelogram ABCD describe a triangle equal in area to ABCD and having the  $\angle$  at A common.

**21.** Construct a triangle of given area with two sides of given length.

**22.** Inscribe a square of given magnitude in a given square.

**23.** Trisect a parallelogram by straight lines drawn through one angular point. [By Ex. 17 trisect the sides of the parallelogram opposite the given angular point.]

**24.** Describe a  $\triangle$  equal to a given quadrilateral. [See Page 68, Ex. 2.]

**25.** Describe a  $\triangle$  equal in area to a given rectilineal figure. [Use Ex. 24, or see page 79.]



## ON LOCI.

**21.** A *locus* consists of all the points which satisfy certain conditions and of those points alone. Thus, for example, the locus of the points in a fixed plane which are at a given distance from a given point is the circumference of the circle described from the given point as centre, with the given distance as radius ; for all the points on the circle, and no others, are at the given distance from the given point.

Again, the locus of all points which are at a given distance from a given straight line is one or other of the two straight lines parallel to the given straight line and at the given distance from the given straight line, one on one side and the other on the other side of it.

We shall restrict ourselves to loci which are situated in a fixed plane, and which are properly called *plane loci*.

Several of the propositions in Euclid furnish good examples of loci. Thus the locus of the vertices of all triangles which are on the same base and on the same side of it, and which have the same area, is a straight line parallel to the base ; this is shewn in I. 37 and I. 39.

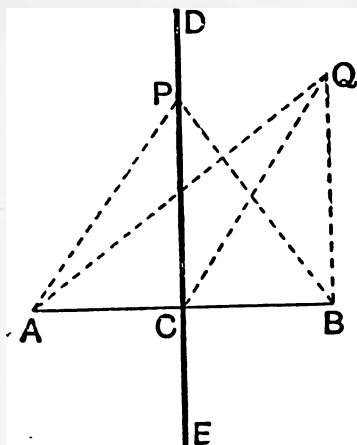
**22.** We will now give some examples. In each example we ought to shew not only that all the points which we indicate as the locus do fulfil the assigned conditions, but that no other points do. This second part however we shall generally leave to the student.

**23.** *Required the locus of points which are equidistant from two given points.*

Let A and B be the two given points. Join AB, and bisect it at C. Draw DCE at right angles to AB. Then DCE shall be the required locus. For take *any* point P on it and join PA and PB. Then in the  $\triangle PCA$ ,  $\triangle PCB$  we have PC common,  $AC=CB$ , and the right  $\angle PCA =$  the right  $\angle PCB$  ;

$\therefore$  the base  $PA =$  the base  $PB$ , and  $\therefore$  P is equidistant from A and B.

Also, no point out of DCE is equidistant from A and B. For, if



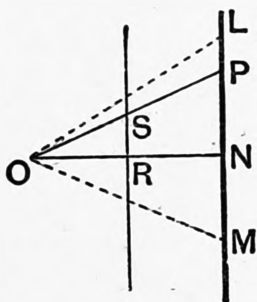
possible, let  $Q$  be such a point. Join  $QA$ ,  $QC$ ,  $QB$ . In the  $\triangle^s$   $QCA$ ,  $QCB$  we then have  $AC = CB$ ,  $QC$  common, and the base  $QA =$  the base  $QB$ ;

$$\therefore \angle QCA = \angle QCB; \quad [\text{I. 8.}]$$

$\therefore$  each is a rt.  $\angle$ ;  $\therefore Q$  lies on the st. line  $DCE$ .

This line is thus the required locus.

**24.** Find the locus of the middle points of the straight lines drawn from a fixed point  $O$  to all points on a given straight line.



Let  $LM$  be the given straight line. Draw  $ON$  perpendicular to it, and bisect  $ON$  at  $R$ . Take *any* point  $P$  on  $LM$ ; join  $OP$ , and bisect it in  $S$ . Join  $RS$ . Since  $R$ ,  $S$  are the middle points of the sides  $ON$ ,  $OP$ ;  $\therefore RS$  is parallel to  $NP$ ; [Art. 1.]

$$\therefore \angle ORS = \angle ONP = \text{a right } \angle; \quad [\text{I. 29.}]$$

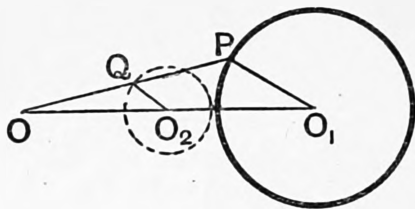
$\therefore S$  lies on the straight line bisecting  $ON$  at right angles.

Similarly, the middle point of the straight line joining  $O$  to any other point of  $LM$  lies on  $RS$ ;

$\therefore$  the required locus is the straight line bisecting  $ON$  at right angles; it is therefore  $\parallel^l$  to  $LM$ .

**25.** *The locus of the middle point of the straight lines drawn from a fixed point O to all points on a given circle is another circle.*

Let the given circle have  $O_1$  as centre. Join O to any point P on the circle. Bisect OP in Q and  $OO_1$  in  $O_2$ . Join  $O_2Q$ .



Since  $O_2$ , Q are the middle points of  $OO_1$  and OP ;

$\therefore O_2Q$  is  $\parallel$  to and one half of  $O_1P$  ;

$\therefore O_2Q$  is constant ; also  $O_2$  is a fixed point ; [Constr.

$\therefore$  Q always lies on a fixed circle whose centre is  $O_2$ , and whose radius is one-half that of the given circle.

**26.** *Required the locus of the vertices of all triangles on a given base AB, such that the square on the side terminated at A may exceed the square on the side terminated at B, by a given square.*

Suppose C to denote a point on the required locus ; from C draw CD perpendicular to AB, produced if necessary.

Then sq. on AC = sqs. on AD, DC,

and sq. on BC = sqs. on BD, DC ; [I. 4.

$\therefore$  sq. on AC - sq. on BC = sq. on AD - sq. on BD ;

that is, given sq. = sq. on AD - sq. on BD ;

$\therefore$  D is a fixed point either in AB or in AB produced through B. [Art. 17. Also CDB is a right  $\angle$ .

The required locus is the straight line drawn through the known point D, at right angles to AB.

## EXERCISES ON LOCI (BOOK I.).

**1.** The locus of points equally distant from two given intersecting straight lines is one or other of two fixed straight lines.

**2.** What is the locus of points equidistant from two parallel straight lines ?

**3.** Find the locus of the middle points of all straight lines parallel to the base of a triangle and terminated by the sides.

**4.** From any point on the base of a triangle straight lines are drawn parallel to the sides ; the locus of the point of intersection of the diagonals of the  $\parallel^{\text{gms}}$  so formed is a straight line parallel to the base.

**5.** The locus of all points, the sum of whose perpendicular distances

from two given perpendicular straight lines is given, is a straight line which is equally inclined to the two given lines.

**6.** Find the locus of all points the difference of whose perpendicular distances from two given straight lines is given.

**7.** Find the locus of a point  $P$  such that the sum of the areas of the triangles  $PAB$ ,  $PCD$  with given bases  $AB$ ,  $CD$  is constant.

[Let  $AB$ ,  $CD$  meet in  $O$ ; take  $E$ ,  $F$  on  $OA$ ,  $OC$  so that  $OE=AB$  and  $OF=CD$ . Then  $\triangle PAB = \triangle POE$  and  $\triangle PCD = \triangle POF$ ;

$\therefore$  area  $OEPF$  is const. But area  $OEF$  is the same for all positions of  $P$ :

$\therefore \triangle EPF$  is of constant area;

$\therefore$  locus of  $P$  is a straight line parallel to  $EF$  by I. 39.]

**8.**  $AB$  and  $CD$  are two straight lines given in magnitude and position, and a point  $P$  moves so that the  $\triangle PAB$ ,  $PCD$  are equal in area; the locus of  $P$  is a straight line passing through the intersection of  $AB$  and  $CD$ , produced if necessary.

[As in the previous solution we have  $\triangle OPE = \triangle OFP$ . Hence if  $OP$  meet  $EF$  in  $N$ , we have  $EN=FN$  (Art. 2);  $\therefore P$  lies on the straight line joining  $O$  and the middle point of  $EF$ .]

**9.** Find the locus of the points of intersection of the diagonals of a parallelogram whose base and area are given.

**10.** Find the locus of the intersection of the medians of all triangles whose base and area are given.

**11.** The locus of the vertices of all right-angled triangles described upon a given straight line as hypotenuse is a circle.

**12.** A ladder is raised gradually, its ends being always in contact with a vertical wall and the ground respectively; the locus of its centre is a portion of a circle.

**13.** The locus of the vertex  $A$  of a triangle  $ABC$ , given the base  $BC$  and the length of the median through  $B$ , is a circle.

[If  $BD$  be the median through  $D$  the locus of  $D$  is a circle, by hypothesis, and  $CA$  always = twice  $CD$ ; then proceed as in Art. 25.]

**14.** Find a point in a given straight line which is equidistant from (1) two given points, and (2) two given straight lines.

**15.** Find a point which is equidistant from three points which are not in the same straight line.

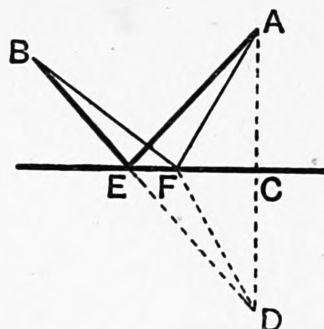
**16.** If two opposite ends of a square of given magnitude move along two straight lines which are at right angles, then the remaining angles move along two perpendicular straight lines.

**17.** Construct a triangle having given the base, the altitude, and the length of the median which bisects the base.

**18.** Construct an isosceles triangle which has the same base and the same area as a given triangle.

## MAXIMA AND MINIMA.

✓ **27.** *In a given indefinite straight line it is required to find a point such that the sum of its distances from two given points on the same side of the straight line shall be the least possible.*



Let A and B be the two given points. From A draw AC perpendicular to the given straight line, and produce AC to D so that  $CD = AC$ . Join DB meeting the given straight line at E.

*Then E shall be the required point.*

For let F be any other point in the given straight line. Then, because  $AC = DC$ , and EC is common to the two  $\triangle^s ACE, DCE$ , and that the right  $\angle ACE =$  the right  $\angle DCE$ ,  $\therefore AE = DE$ . [I. 4.]

Similarly,  $AF = DF$ . Also the sum of DF and FB  $> BD$ . [I. 20.]  $\therefore$  the sum of AF and FB  $> BD$ ;

that is, the sum of AF and FB  $>$  the sum of AE and EB.

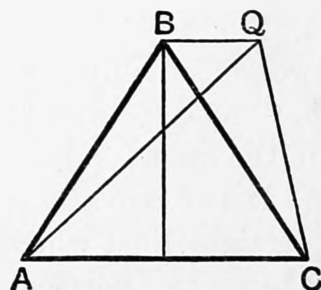
**Cor.** Since the  $\triangle^s AEC, CED$  are equal in all respects, the  $\angle AEC = \angle DEC$ , and  $\therefore$  the point E is such that the sum of its distances from A, B is a minimum when these distances are equally inclined to the given straight line.

✓ **28.** *The perimeter of an isosceles triangle is less than that of any other triangle of equal area standing on the same base.*

Let ABC be an isosceles triangle; QAC any other triangle equal in area and standing on the same base AC.

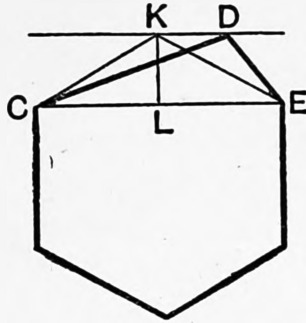
Join AQ; then BQ is parallel to AC. [I. 39.]

Since AB, BC are equally inclined to AC, and  $\therefore$  to BQ, it follows, from Art. 27 Cor., that the sum of AB, BC is always less than the sum of AQ, QC.



**Cor.** Of all  $\triangle^s$  of the same perimeter that which has the greatest area is isosceles.

29. If a polygon be not equilateral a polygon may be found of the same number of sides, and equal in area, but having a less perimeter.

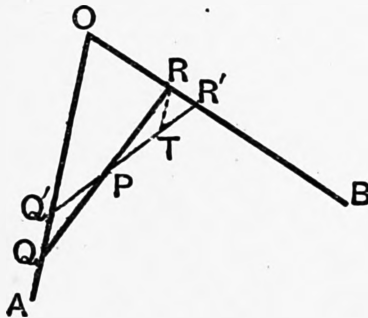


For, let CD, DE be two adjacent unequal sides of the polygon. Join CE. Through D draw a straight line parallel to CE. Bisect CE at L; from L draw a straight line at right angles to CE to meet the straight line drawn through D at K. Then by removing from the given polygon the triangle CDE and adding the triangle CKE, we obtain a polygon having the same number of sides as the given polygon, and equal to it in area, but having a less perimeter. [Art. 28.]

30. Through a given point P draw a straight line which shall cut off from two given straight lines OA, OB a triangle of minimum area, and shew that the required straight line is bisected at P.

Draw through P a straight line QR which is bisected at P.

[Art. 18.]



Then QOR shall be the required minimum  $\Delta$ .

For draw any other line Q'PR' to cut off the  $\Delta$  OQ'R'.

Draw RT parallel to OQ' to meet Q'PR' in T.

Then  $PR = PQ$ ,  $\angle TPR = \angle Q'PQ$ , and  $\angle PRT = \angle PQQ'$ .

[I. 29.]

$$\therefore \Delta PQQ' = \Delta PTR.$$

$$\therefore \Delta PRR' > \Delta PQQ'.$$

Add to each the area ROQ'P.

$$\therefore \Delta Q'OR' > \Delta QOR.$$

$\therefore$  any other straight line through P cuts off a greater  $\Delta$  than QR cuts off.

$\therefore$  QOR is the required minimum  $\Delta$ .

In this question it will be noticed that the position of QR, when it cuts off the minimum area, is a **symmetrical** one, in that it is bisected at P. We shall find that this characteristic of being a symmetrical position is a general property of lines, areas, and angles, which are either maxima or minima.

Thus in Art. 27, if F is to the right of E, the  $\angle AFC > \angle BFE$ ; if it be to the left, then  $\angle AFC < \text{the corresponding angle}$ ; it is only when F is at the critical, or turning, point E that these angles are just equal.

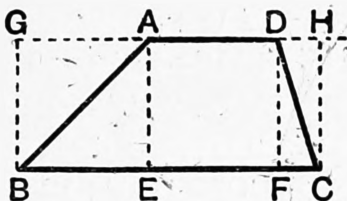
Similarly in Art. 28, if Q be anywhere on AQ to the right or the left of A, the sides AQ, QC are not equal, they are only equal when Q is at the critical point B.

### EXERCISES.

1. Find the triangle of greatest area, two of its sides being given.
2. Find the parallelogram of greatest area, two of its adjacent sides being given.

## THEOREMS AND EXAMPLES ON BOOK II.

31. *The area of a trapezium equals half the rectangle contained by the sides and the perpendicular distance between them.*



Let ABCD be the trapezium with AD, BC the parallel sides. Draw perp<sup>s</sup> AE, DF, BG, CH on the opposite sides.

Then  $\triangle ABE = \text{half the rect. EG}$ ,

and  $\triangle DFC = \text{half the rect. FH}$ .

[I. 41.]

$\therefore$  twice  $\triangle ABE + \text{twice } \triangle DFC + \text{twice rect. AF}$   
 $= \text{rect. EG} + \text{rect. FH} + \text{twice rect. AF}$   
 $= \text{rect. BH} + \text{rect. AF} = \text{rect. AE, BC} + \text{rect. AD, AE}$ .

that is, area ABCD

$= \text{half the rect. contained by AE and the sum of BC, AD.}$

32 *If ABC be a triangle, D the middle point of BC and E the foot of the perpendicular from A upon BC, the difference of the squares on AB, AC = twice the rect. BC, DE.*

We have  $AB^2 = AE^2 + BE^2$ ,

and  $AC^2 = AE^2 + CE^2$ ,

$\therefore$  the diff. of the sqs. on AB, AC  
 $= \text{the diff. of the sqs. on BE, EC}$   
 $= \text{the rect. contained by the sum and difference of BE, EC.}$

[II. 5, Note.]

Now the sum of BE, EC = BC, and

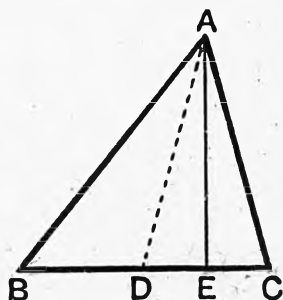
their difference =  $BE - EC = BE + DE - DC = BD + 2DE - DC$   
 $= 2DE$ , since D is the middle pt. of BC.

$\therefore$  diff. of sqs. on AB, AC = 2 rect. BC, DE.

This proposition may be enunciated thus: *The difference of the squares on the sides of a  $\triangle$  = twice the rect. contained by the base and the projection on the base of the median through the vertex.*

It has already been shewn [II. 13, Exercise I.] that

$$AB^2 + AC^2 = 2AD^2 + 2DB^2.$$





**33.** *ABCD is any quadrilateral and P, Q the middle points of its diagonals AC, BD; then the sum of the squares on its sides exceeds the squares on its diagonals by four times the square on the line joining the middle points of the diagonals, i.e.*

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

For [by the previous theorem]

$$AB^2 + BC^2 = 2AP^2 + 2BP^2,$$

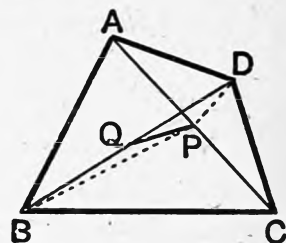
$$\text{and } AD^2 + DC^2 = 2AP^2 + 2DP^2.$$

$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = 4AP^2 + 2BP^2 + 2DP^2 \\ = AC^2 + 2BP^2 + 2PD^2.$$

$$\text{Also } BP^2 + PD^2 = 2PQ^2 + 2DQ^2$$

$$\therefore 2BP^2 + 2PD^2 = 4PQ^2 + 4DQ^2 = 4PQ^2 + BD^2.$$

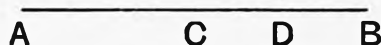
$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$



In the particular case when ABCD is a  $\parallel^{\text{gm}}$  P and Q coincide, and  $\therefore$  PQ vanishes. The proposition then is; *The sum of the sqs. on the sides of a parallelogram = sum of the sqs. on the diagonals.*

**34.** *If a straight line AB is divided into two parts at D,*

(1) *the sum of the squares on AD, DB is least,*  
and (2) *the rect. AD, DB is greatest, when the straight line is bisected.*



Let C be the middle point of AB.

(1) By II. 9 we have  $AD^2 + DB^2 = 2AC^2 + 2CD^2$ .

Now AC is of the same length wherever the point D is.

$\therefore 2AC^2 + 2CD^2$  is least when CD is least, and this is when CD is zero, i.e. when D coincides with C.

$\therefore AD^2 + DB^2$  is least when D is at C.

(2) By II. 5 we have  $\text{rect. AD, DB} + CD^2 = AC^2 = \text{a constant for all positions of D.}$

$\therefore$  rect. AD, DB is greatest when  $CD^2$  is least, i.e. when CD is zero, i.e. when D coincides with C.

**Corollary.** Since, by (2), the rectangle contained by two straight lines, AD, DB, whose sum is given, is greatest when they are equal, it follows that *Of all rectangles with a given perimeter the greatest is a square.*

**EXERCISES.**

**\*\*1.** The locus of the vertex of a triangle on a given base, when the sum of the squares on its sides is given, is a circle.

[For, in the figure of Art, 32, DA is constant and known.]

**\*\*2.** When the difference of the squares is given, the locus is a straight line.

[For in this case DE is constant and known.]

**\*\*3.** The base BC of a triangle ABC is divided at D, so that

$$m \cdot BD = n \cdot CD;$$

prove that

$$m \cdot AB^2 + n \cdot AC^2 = m \cdot BD^2 + n \cdot DC^2 + (m + n) AD^2.$$

If D be in BC produced, then

$$m \cdot AB^2 - n \cdot AC^2 = m \cdot BD^2 - n \cdot DC^2 + (m - n) \cdot AD^2.$$

[Use II. 12 and 13.]

**\*\*4.** Given the base of a  $\triangle$  in magnitude and position, and the sum, or difference, of  $m$  times the square on one side and  $n$  times that on the other, prove that the locus of the vertex is a circle.

[Use the previous Exercise.]

**\*\*5.** If the medians of a triangle ABC meet in G, prove that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

**\*\*6.** If G be the centroid of a  $\triangle ABC$  and P any point in its plane, prove that  $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3 \cdot PG^2$ . [Use Ex. 3.]

**7.** The locus of a point, which moves so that the sum of the squares of its distances from the angular points of a triangle is constant, is a circle whose centre is the centroid of the triangle. [Use Ex. 6.]

**8.** Find a point in the plane of a triangle such that the sum of the squares of its distances from the angular points is a minimum.

[Use Ex. 6.]

**9.** Find a point in a given straight line such that the sum of the squares of its distances from two given points is a minimum.

Divide a given straight line AB at C so that

**10.**  $AB \cdot BC = \text{given square}$ . [Use I. 44.]

**11.**  $AC^2 = 2BC^2$ .

**12.**  $AB^2 + BC^2 = 2AC^2$ .

Produce a given straight line AB to C, so that

**13.**  $AB^2 + AC^2 = 2AC \cdot BC$ .

**14.**  $AB^2 + BC^2 = 2AC \cdot BC$ .

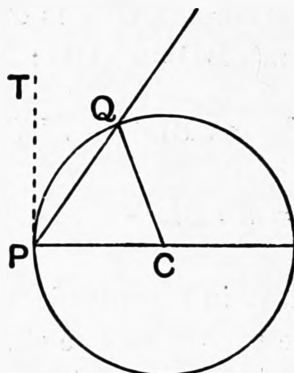
**15.**  $AC \cdot BC = AB^2$ . [BC = AH, Fig. II., 11.]

**16.**  $AC \cdot AB = \text{a given square}$ .

## THEOREMS AND EXAMPLES ON BOOK III.

## DEFINITION OF A TANGENT AS A LIMIT.

✓ **35.** Another definition of the tangent is often much more useful than Euclid's.



If PQ be a straight line through a point P on a circle, of centre C, which meets it again in Q, then PQ is called a **secant**.

If P be kept fixed and Q move along the circle until it coincides with P the limiting position of PQ, when Q becomes indefinitely close to P, is called the **tangent** at P.

[If the student conceive the circumference of the circle as made up of infinitely small dots, packed infinitely close together, then this method of looking upon the tangent at P conceives it as the line joining P and the very next dot to P.]

**36.** We can easily shew that the tangent PT is perpendicular to PC. For since  $CQ = CP$ , the  $\angle CPQ = \angle CQP$ .

$$\therefore \text{twice } \angle CPQ = \angle CPQ + \angle CQP = 2 \text{ rt. } \angle^s - \angle PCQ. \quad [\text{I. 32.}]$$

This holds for all positions of Q. When Q is indefinitely close to P, the  $\angle PCQ$  vanishes, and then in the limiting position twice the  $\angle CPQ$  is two right  $\angle^s$ .

But in the limit the  $\angle CPQ$  becomes the  $\angle CPT$ ,

$$\therefore \text{twice the } \angle CPT = \text{two right } \angle^s,$$

$$\text{i.e. } \angle CPT = \text{a right } \angle.$$

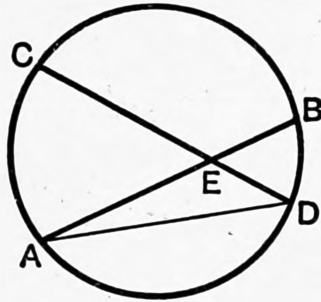
We can also deduce III. 32. For let Q be a point close to B, in the figure of that proposition, and produce BQ to R. Then BQDA is a cyclic quadrilateral, and

$$\begin{aligned} \therefore \angle BAD &= \text{two rt. } \angle^s - \angle BQD \\ &= \angle DQR. \end{aligned} \quad [\text{III. 22.}]$$

Now let  $Q$  become indefinitely close to  $B$ , then  $BQR$  ultimately coincides with  $BF$ , and

$\therefore \angle BAD = \text{the ultimate position of } \angle DQR = \angle DBF$

✓ **37.** *If two chords intersect within a circle, the angle which they include is measured by half the sum of the intercepted arcs.*



Let the chords  $AB$  and  $CD$  intersect at  $E$ ; join  $AD$ .

The  $\angle AEC = \text{the } \angle^s \text{ ADE, DAE,}$

[I. 32.

that is,  $= \text{the } \angle^s \text{ standing on the arcs AC, BD,}$

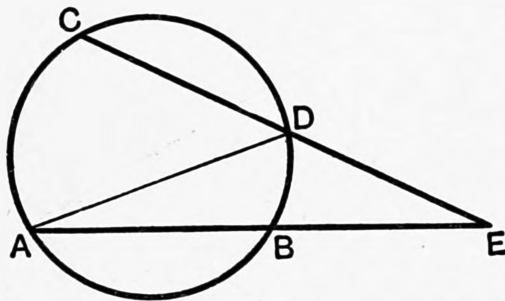
that is,  $= \text{the } \angle \text{ at the circumference standing on the sum of the arcs AC, BD,}$

that is,  $= \text{the } \angle \text{ at the centre standing on half the sum of the arcs AC, BD.}$

[III. 20.

Similarly the angle  $CEB$  is measured by half the sum of the arcs  $CB$  and  $AD$ .

✓ **38.** *If two chords produced intersect without a circle, the angle which they include is measured by half the difference of the intercepted arcs.*



Let the chords  $AB$  and  $CD$ , produced, meet in  $E$ ; join  $AD$ .

The  $\angle ADC = \text{the } \angle^s \text{ EAD, AED,}$

[I. 32.

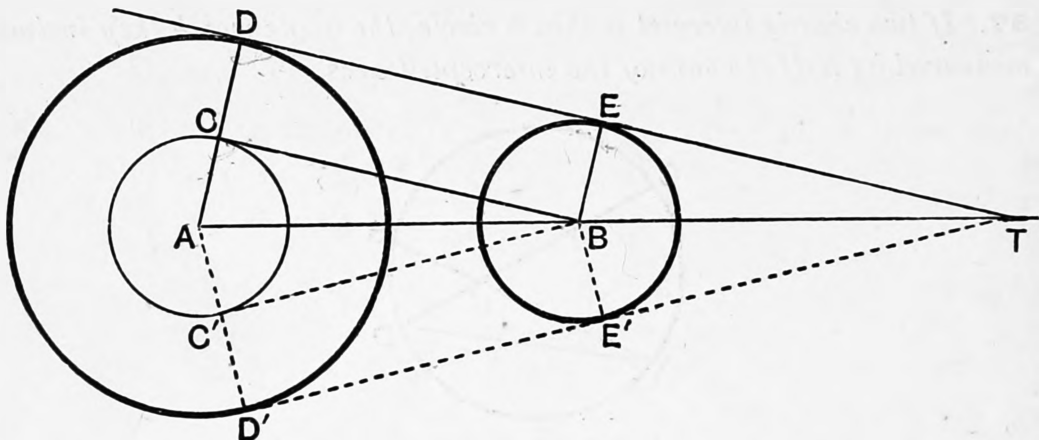
that is,  $\angle AEC = \text{difference of the } \angle^s \text{ ADC, BAD,}$

that is,  $= \angle \text{ at circumference standing on the difference of the arcs AC, BD,}$

and  $\therefore = \angle \text{ at centre standing on half the difference of these arcs.}$

✓ **39.** *To draw a common tangent to two given circles.*

Let A be the centre of the greater circle, and B the centre of the less circle. With centre A, and radius equal to the difference of the



radii of the given circles, describe a circle ; from B draw a tangent BC to the circle so described.

Join AC and produce it to meet the circumference at D.

Draw the radius BE parallel to AD, and on the same side of AB ; and join DE.

*Then DE shall touch both circles.*

For BE and CD are equal and parallel lines.

[Construction.

$\therefore$  BC and DE are equal and parallel.

[I. 33.

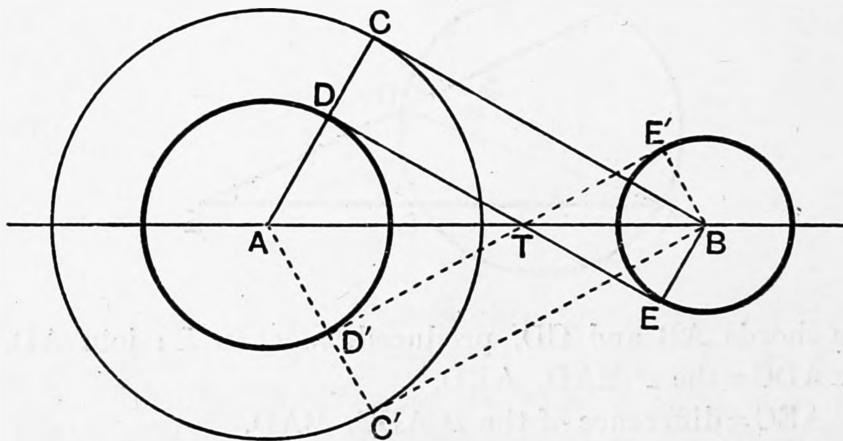
$\therefore \angle ADE = \angle ACB$  [I. 29] = a right  $\angle$ .

$\therefore$  since BEDC is a  $\parallel^m$ ,

$\angle BED = \angle BCD$  = a right  $\angle$ .

$\therefore$  DE touches both circles at D, E.

[III. 16, Corollary.



Since two tangents can be drawn from B to the described circle, two solutions can be obtained ; and the two straight lines which are thus

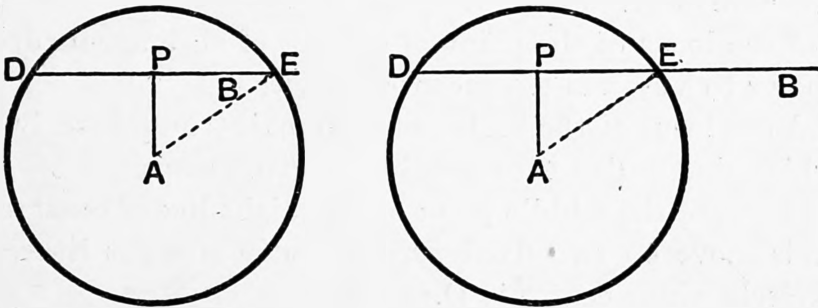
drawn to touch the two given circles can be shewn to meet AB, produced through B, at the same point T. The construction is applicable when each of the given circles is without the other, and also when they intersect.

When each of the given circles is without the other, we can obtain two other solutions. For, as in the second figure, describe a circle with A as centre and radius equal to the sum of the radii of the given circles ; and continue as before, except that BE and AD will now be on *opposite* sides of AB. The two straight lines which are thus drawn to touch the two given circles can be shown to intersect AB at the same point. [See Art. 84.]

### LOCI [BOOK III.].

**40.** *Required the locus of the middle points of all the chords of a circle which pass through a fixed point.*

Let A be the centre of the given circle ; B the fixed point ; let any



chord DE of the circle be drawn so that, produced if necessary, it may pass through B. Let P be the middle point of this chord, so that P is a point on the required locus.

Since P is the middle point of the chord DE,

$\therefore \angle APE$ , that is  $\angle APB$ , is a right  $\angle$ .

[III. 3.]

$\therefore$  P is on a circle of which AB is a diameter. Hence if B be within the given circle the locus is the circle described on AB as diameter ; if B be without the given circle the locus is that part of the circle described on AB as diameter, which is within the given circle.

✓ **41.** *A chord of given length slides round with its ends on the circumference of a given circle ; the loci of*

- (1) *its middle point,*
- (2) *any fixed point on it, are circles.*

Let DE [Fig. Art. 40] be the chord of given length, P its middle point ; A the centre.

Then  $\angle APE$  is a rt.  $\angle$ .

[III. 3.]

$\therefore AP^2 = AE^2 - PE^2 = a$  constant, since the radius of the circle and the length of the chord are given.

$\therefore AP$  is constant, and hence P moves on a circle whose centre is A.

(2) If B be the fixed point on the chord, then PB is constant.

$\therefore AB^2 = AP^2 + PB^2 = \text{constant}.$

$\therefore AB$  is constant, and the locus of B is a circle whose centre is A.

## EXERCISES.

1. Find the locus of the middle points of parallel chords of a circle.
2. Find the locus of the middle points of equal chords of a circle.
3. Find the locus of the point of intersection of tangents to a given circle which meet at a given angle.
4. Find the locus of all points from which tangents to a given circle are equal.
5. Find the locus of the points from which the tangents drawn to a given circle are of given length.
6. Find the locus of the point of contact of all tangents drawn from a given point to a system of concentric circles.
7. Find the locus of the vertex of a triangle whose base BC is given and of which the median through B is of given length.
8. The locus of the middle point of a straight line of constant length, whose ends move on two fixed perpendicular straight lines, OA and OB, is a circle whose centre is O.
9. Given the base and vertical angle of a triangle ; find the locus of the middle point of the straight lines joining the vertices of all such triangles to the middle point of the base.
10. Given the base BC of a triangle and its vertical angle, prove that the loci of
  - (1) the intersection of the bisectors of the base  $\angle$ 's,
  - (2) the intersection of the bisectors of the exterior base  $\angle$ 's,
  - (3) the intersection of the bisector of one exterior base  $\angle$  with the bisector of the other interior base  $\angle$ ,
 are all arcs of circles passing through B and C.



**11.** If two segments of circles have a common chord AB and any two points P and Q be taken, one on each segment, prove that the locus of O, the point of intersection of the bisectors of the angles PAQ, PBQ, is another segment of a circle passing through A and B.

$$[\text{Twice } \angle AOB = 4 \text{ rt. } \angle^s - 2 \angle OAB - 2 \angle OBA$$

$$= 4 \text{ rt. } \angle^s - \angle PAB - \angle QAB - \angle PBA - \angle QBA$$

$$= \angle APB + \angle AQB = \text{const. } \therefore \text{etc.}]$$

**12.** If A and B be two fixed points on a circle and C, D the extremities of a chord of constant length, then the intersection of AD, BC and also that of AC, BD lie on fixed circles.

**13.** O is a fixed point on the circumference of a circle and OP is any chord. On OP is described a circle containing a given angle; the locus of its centre is one or other of two fixed circles.

**14.** PQ is a straight line of given length which moves so that its ends, P and Q, slide on two given straight lines OA and OB; the locus of the intersection of perpendiculars to OA and OB, drawn through P and Q respectively, is a circle.

**15.** If a chord of a circle, centre O, subtend a right angle at a fixed point P, the locus of its middle point is a circle whose centre is the middle point of PO.

[If QR be the chord, and C, T be the middle points of OP, QR then by Page 109, Ex. 1,  $4OC^2 + 4CT^2 = 2OT^2 + 2TP^2 = 2OQ^2 - 2TR^2 + QP^2 + PR^2 - 2TR^2 = 2OQ^2 - 4TR^2 + QR^2 = 2OQ^2$ , etc.]

Find the locus of the centres of all circles which

**16.** pass through two given points.

**17.** touch two given intersecting straight lines.

**18.** touch a given straight line at a given point.

**19.** touch a given straight line and are of given radius.

**20.** touch a given circle at a given point.

**21.** touch a given circle and are of given radius.

**22.** A triangle is formed by a fixed tangent to a circle, a variable tangent, and the chord joining their points of contact. Find the locus of the centre of its circumscribing circle.

**23.** Tangents are drawn, one to each of two concentric circles, and the angle between them is constant. Prove that the locus of their point of intersection is one or other of two fixed circles concentric with the given one.

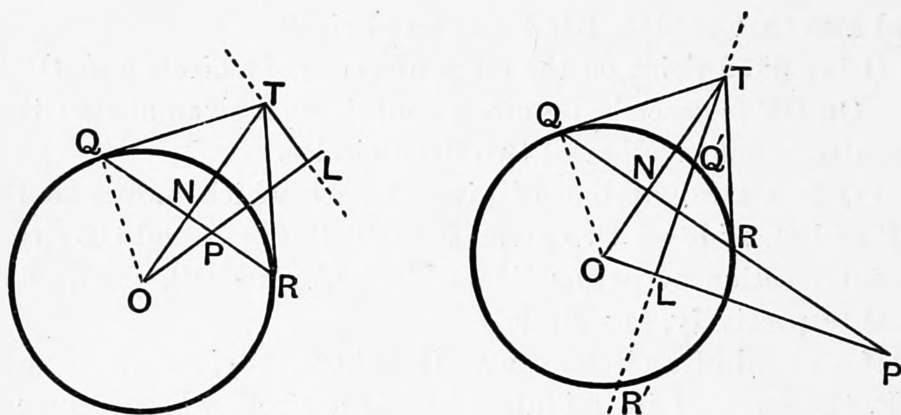
**24.** Two adjacent corners A, B of a sheet of paper are doubled down and meet at a point P in such a manner that the three parts of the edge AB form a triangle right-angled at P. Prove that the locus of P is an arc of a circle.



## POLE AND POLAR.

**42.** The Polar of any point  $P$  with respect to a circle is the locus of the point of intersection of tangents drawn at the extremities of any chord of the circle which passes through  $P$ .

*To find the polar of any given point  $P$  and to construct it geometrically.*



Through  $P$  draw any straight line to cut the circle in  $Q$  and  $R$ . At  $Q$  and  $R$  draw tangents and let them meet in  $T$ . [III. 17.] Join  $OP$  and draw from  $T$  a perpendicular  $TL$  on  $OP$  or  $OP$  produced. Join  $OT$  and let it meet  $QR$  in  $N$ .

Then  $OQT$  is a  $\triangle$  having  $OQT$  a right  $\angle$ , and  $QN$  is perpendicular to the base  $OT$ ;

$$\therefore ON \cdot OT = OQ^2 = (\text{radius})^2.$$

[For, by III. 13,  $TQ^2 = TO^2 + OQ^2 - 2 ON \cdot OT$ ;

$$\therefore 2 ON \cdot OT = TO^2 + OQ^2 - TQ^2 = 2 OQ^2 \text{ (I. 47).}]$$

But since  $TNP$ ,  $TLP$  are both right angles, a circle will go round  $T$ ,  $N$ ,  $P$ ,  $L$ , and

$$\therefore OP \cdot OL = ON \cdot OT; \quad [\text{III. 36.}]$$

$$\therefore OP \cdot OL = (\text{radius})^2;$$

$\therefore OL$  is constant, since  $P$  is a given point, and therefore  $OP$  constant;

$\therefore L$  is a fixed point and  $\angle OLT$  is a right angle;

$\therefore$  the locus of  $T$  (that is, the polar of  $P$ ) is a straight line perpendicular to  $OL$  and passing through  $L$ .

The polar of  $P$  is thus a straight line, and may be constructed as follows:

*On  $OP$  take a point  $L$ , such that  $OL \cdot OP = (\text{radius})^2$ , and through  $L$  draw a straight line perpendicular to  $OP$ .*

The two points,  $L$  and  $P$ , are often called **inverse points**.

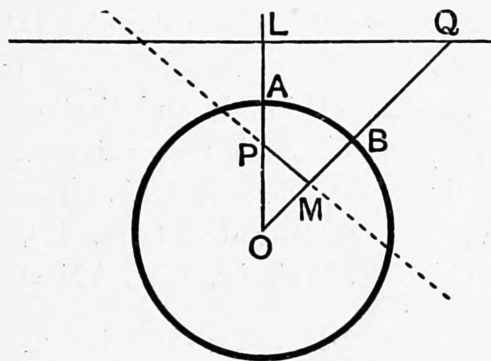
**Corollary 1.** If P be on the circle so also is L, and the polar of P in this case is the tangent at P.

**Corollary 2.** If, as in the right-hand figure, the line TL meet the circle in Q' and R' then the straight lines PQ', PR' would be tangents at Q', R'.

For  $OL \cdot OP = (\text{radius})^2$ , and  $Q'LR'$  is perpendicular to OP.

Hence, if the point P lie outside the circle, its polar coincides with the line joining the points of contact of tangents drawn from it to the circle.

**43.** *If the polar of a point P passes through Q then the polar of Q passes through P.*



Let Q be any point on the polar LQ of a point P. Draw OPL perpendicular to LQ, O being the centre.

Then  $OP \cdot OL = (\text{radius})^2$ . [Art. 42.]

Join OQ and draw PM perpendicular to OQ.

Then since PLQ, PMQ are right angles, a circle will pass through P, L, Q, M;

$\therefore OM \cdot OQ = OP \cdot OL$  [III. 36.]  
 $= (\text{radius})^2$ .

Hence, since OMP is a right angle, it follows, by Art. 42, that PM is the polar of Q, that is, *the polar of Q passes through P.*

*The polars of P and Q meet at an angle equal to that subtended by P and Q at the centre of the circle.*

For, if O be the centre, the polar of P is perpendicular to OP, and the polar of Q is perpendicular to OQ.

Also, the  $\angle$  between two straight lines is equal to the angle between their perpendiculars;

$\therefore$  the  $\angle$  between the polars of P and Q = the  $\angle POQ$ .

## EXERCISES.

**\*\*1.** The straight line joining two points, A and B, is the polar of the point of intersection of the polars of A and B.

[Let the latter intersect at T; then A lies on the polar of T, since T lies on the polar of A (43); so B lies on the polar of T;  $\therefore$  AB is the polar of T.]

**\*\*2.** The point of intersection of any two straight lines is the pole of the straight line joining their poles.

**\*\*3.** Find the locus of the poles of all straight lines which pass through a given point. [Use Art. 43.]

**\*\*4.** A and B are two points in a plane of a circle whose centre is C; AX and BY are the perpendiculars from A and B on the polars of B and A respectively; prove that the rectangles CA . BY and CB . AX are equal. [Salmon's Theorem.]

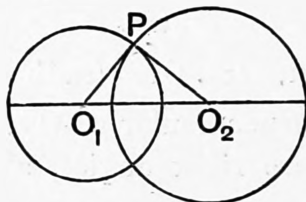
[Let the polar of A meet CA in M, and that of B meet CB in N; also draw AU, BV  $\perp$  CB, CA respectively.

Then A, V, U, B lie on a circle;  $\therefore$  CU . CB = CA . CV. [III. 36.

But CN . CB = CM . CA = sq. on radius (Art. 42). Subtract;

$\therefore$  CB . NU = CA . MV, i.e. CB . AX = CA . BY.]

**44. Orthogonal Circles.** Def. Two circles are said to intersect orthogonally when the tangents at their points of intersection are at right angles.



If the two circles intersect at P, the radii  $O_1P$  and  $O_2P$ , which are perpendicular to the tangents at P, must also be at right angles.

Hence  $O_1O_2^2 = O_1P^2 + O_2P^2$ ,

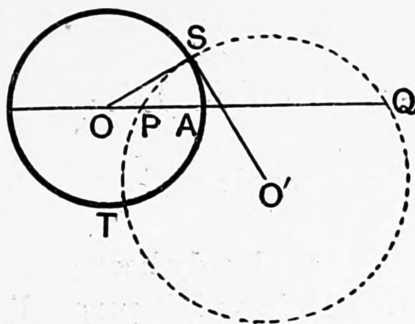
i.e. the square of the distance between the centres must be equal to the sum of the squares of the radii.

Also the tangent from  $O_2$  to the other circle is equal to the radius  $a_2$ , i.e. if two circles be orthogonal the length of the tangent drawn from the centre of one circle to the second circle is equal to the radius of the first.

Either of these two conditions will determine whether the circles are orthogonal.

It follows that if we want the circle whose centre is  $O_2$  which shall cut a given circle, centre  $O_1$ , orthogonally, we must take its radius equal to the tangent from  $O_2$  to the given circle.

**45.** *Given a circle and two inverse points, P and Q, with respect to it; any circle which passes through P and Q cuts the given circle orthogonally.*



Let any such circle cut the given circle in S and T. Let O, O' be the centres of the two circles and join OS, O'S. Then, by hypothesis,  $OP \cdot OQ = OA^2 = OS^2$ .

$\therefore$  OS touches the second circle at S.

[III. 37.]

$\therefore$  OSO' is a right angle.

$\therefore$  the two circles cut orthogonally at S and similarly at T.

### EXERCISES.

**\*\*1.** What is the locus of the centres of all circles which cut a given circle orthogonally at a given point?

**\*\*2.** Through a given point draw a circle to cut a given circle orthogonally at a given point.

**\*\*3.** If two circles cut orthogonally the extremities, P and Q, of any diameter of either, are conjugate points with respect to the other, i.e. the polar of P passes through Q.

[Let the centres of the circles be O and O', and let them cut in C; let PO'Q be any diameter of the second; join OP and draw QS perp<sup>r</sup> to OP; then S lies on the second circle. Since the circles cut orthogonally the tangent at C to the circle O' passes through O.

$\therefore OS \cdot OP = OC^2 = \text{sq. on radius of first circle};$

$\therefore$  since PSQ is a right  $\angle$ , SQ is the polar of P with respect to the circle O (Art. 42);  $\therefore$  etc.]

**4.** P is any point in the plane of a circle C and Q any point on its polar with respect to C; the circle on PQ as diameter cuts C orthogonally.

**5.** All circles which pass through a given point and cut a given circle orthogonally pass through another given point.

[This is the converse of Art. 45.]

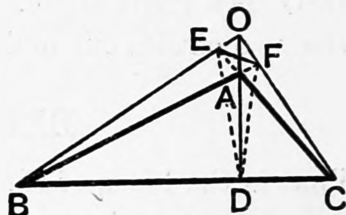
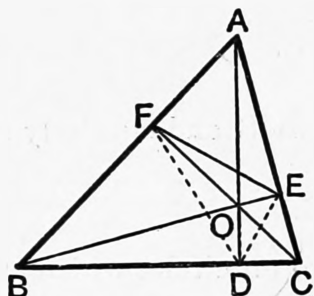
**6.** The chords joining two fixed points on a circle to the ends of any diameter intersect on a fixed circle which cuts the former orthogonally.

## BOOK IV.

**46.** *The perpendiculars from the angles of a triangle on the opposite sides meet in a point.*

[A proof of this theorem has already been given in Art. 8].

Let  $ABC$  be a  $\triangle$ ; from  $B$  draw  $BE$  perpendicular to  $CA$  and  $CF$  perpendicular to  $AB$ ; let  $BE$ ,  $CF$  meet in  $O$ ; join  $AO$  and produce it to meet  $BC$  in  $D$ ; then  $AD$  shall be perpendicular to  $BC$ .



[If the  $\triangle$  be obtuse-angled, as in the second figure, some of these perpendiculars will meet the opposite sides produced].

Since  $\angle AEO$ ,  $\angle AFO$  are right  $\angle$ 's a circle will pass through  $A$ ,  $E$ ,  $O$ ,  $F$ .

[III. 22, *Converse*.  
[III. 21.]

$$\therefore \angle FAO = \angle FEO.$$

Similarly, a circle will pass through  $B$ ,  $C$ ,  $E$ ,  $F$ .

$$\therefore \angle FEO = \angle FCB;$$

$$\therefore \angle FAO = \angle FCB, \text{ that is, } \angle BAD = \angle FCB.$$

Also the  $\angle$  at  $B$  is common to the  $\triangle$ 's  $BAD$ ,  $BCF$ .

$$\therefore \text{the third } \angle BDA = \text{the third } \angle BFC$$

[I. 32.]

$$= \text{a right } \angle.$$

[*Construction*.]

$$\therefore AD \text{ is perpendicular to } BC.$$

**47.** *When  $ABC$  is an acute-angled triangle the angles of the triangle  $DEF$  in the previous figure are equal to the supplements of twice the angles of the original triangle  $ABC$ , and its sides are equally inclined to the sides of the triangle  $ABC$ .*

Since  $O$ ,  $F$ ,  $A$ ,  $E$  lie on a circle,

$$\therefore \angle OFE = \angle OAE = \text{a right } \angle - \angle ACD.$$

[III. 21.]

So, since  $O$ ,  $F$ ,  $B$ ,  $D$  lie on a circle,

$$\therefore \angle OFD = \angle OBC = \text{a right } \angle - \angle ECB.$$

$\therefore$  OFE and OFD are equal angles, and their sum  
= two right  $\angle^s - 2\angle ACB$ .

that is,  $\angle DFE =$  the supplement of twice the  $\angle C$ .

So the angles FDE, DEF are the supplements of twice the angles A and B.

Also since  $\angle EFO = \angle DFO$ ,  $\therefore \angle EFA = \angle DFB$ .

$\therefore$  FE, FD are equally inclined to AB.

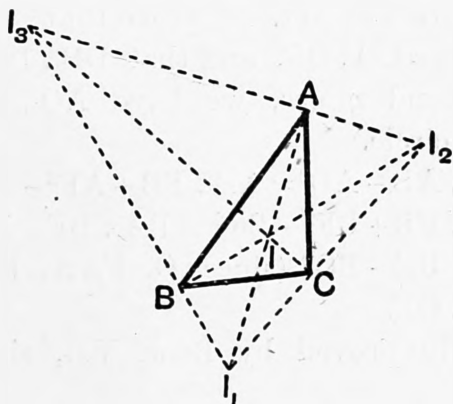
Similarly FE, ED are equally inclined to AC, and ED, DF are equally inclined to BC.

**Definition.** The triangle DEF is often called the **Pedal Triangle** of ABC.

**48.** If I be the in-centre and  $I_1, I_2, I_3$  the three e-centres of a triangle ABC, then

(1) I is the orthocentre of the triangle  $I_1I_2I_3$ .

(2) ABC is the pedal triangle of  $I_1I_2I_3$ .



For AI bisects the  $\angle BAC$ , since AB, AC are tangents to a circle centre I. [III. 17, Cor. 2.]

For a similar reason  $AI_1$  bisects the same  $\angle$ .

$\therefore$   $AI, AI_1$  is a straight line, and similarly  $BII_2, CII_3$ .

Also since BA produced and AC touch a circle centre  $I_2$ .

$\therefore$   $AI_2$  bisects the exterior angle of BAC. [III. 17, Cor. 2.]

Also AI bisects  $\angle BAC$ .

$\therefore$   $\angle IAI_2$  is a right angle. Similarly  $\angle IAI_3$  is a right  $\angle$ .

$\therefore$   $I_2AI_3$  is a straight line. Similarly  $I_3BI_1$  and  $I_1CI_2$  are straight lines.

$\therefore$   $I_1IA, I_2IB$ , and  $I_3IC$  are the perpendiculars from  $I_1, I_2, I_3$  on the sides of the  $\triangle I_1I_2I_3$ .

$\therefore$  I is the orthocentre and ABC the pedal  $\triangle$  of the  $\triangle I_1I_2I_3$ .

## EXERCISES.

**\*\*1.** The perpendicular from the middle point of the side of any triangle on the opposite side of its pedal triangle bisects the latter.

**\*\*2.** O is the orthocentre of a triangle ABC and the perpendicular AD on BC, when produced, meets the circumcircle in H; prove that  $DH = DO$ .

[For  $\angle DCH = \angle DAB$  (III. 21) = rt.  $\angle$  -  $\angle ABD = \angle BCF$ .

$\therefore \triangle^s$  OCD, HCD are equal in all respects;  $\therefore$  etc.]

**\*\*3.** If O is the orthocentre of the triangle ABC, then either of the four points O, A, B, C is the orthocentre of the triangle formed by the other three.

**\*\*4.** If ABC be a triangle obtuse-angled at A, and AD, BE, CF be the perpendiculars on the sides, then BE, CF bisect the exterior angles of the  $\triangle DEF$ , and AD bisects the interior angle.

**5.** The radii from the circumcentre of a triangle to the angular points are respectively perpendicular to the straight lines joining the feet of the perpendiculars on the sides of the triangle from the opposite vertices.

**\*\*6.** With the letters of Art. 46, prove that

$AO \cdot OD = BO \cdot OE = CO \cdot OF$ , and that  $DB \cdot DC = DO \cdot DA$ , etc.

[Taking the left-hand figure, we have  $AO \cdot AD = AF \cdot AB$ , since F, B, D, O lie on a circle.

$$\begin{aligned} \therefore AO \cdot OD &= AF \cdot AB - AO^2 = AF \cdot FB + AF^2 - AO^2 = AF \cdot FB - OF^2 \\ &= AF \cdot FB + BF^2 - BO^2 = BA \cdot BF - BO^2 \\ &= BO \cdot BE - BO^2 \text{ (since O, F, A, E lie on a circle)} \\ &= BO \cdot OE. \end{aligned}$$

This is more easily proved by Book VI., since EOA, DOB are similar  $\triangle^s$ ].

**\*\*7.** In the second figure of Art. 46, if with centre O and radius, whose square is equal to either of the rectangles  $OA \cdot OD$ ,  $OB \cdot OE$ ,  $OC \cdot OF$ , a circle be described, then each angular point of the triangle ABC is the polar with respect to this circle of the opposite side.

[This follows from the last Exercise and Art. 42. Such a circle is called the **Polar Circle** of the triangle; it is also called the **Self-Conjugate Circle**. If the  $\triangle$  be acute-angled, as in the first figure of Art. 46, there is no such circle. For its centre, if any, must, by Art. 42, lie on each of the three lines AD, BE, and CF, and must therefore be at O. But since in this case the point A and the line BC lie on **opposite** sides of O, it is impossible by Art. 42 that A should be the polar of BC.]

**8.** The circles described on the sides of a triangle as diameters cut the polar circle orthogonally.

[Taking the right-hand figure of Art. 46, the circle on AC as diameter passes through A and D, which are inverse points with respect to the polar circle, since  $OA \cdot OD = \text{sq. on radius of the polar circle}$ . Then apply Art. 45.]

**9.** Find a point O within or without a triangle ABC such that the circumcircles of the triangles OAB, OBC, OCA are all equal.

**10.** Prove that the orthocentre and vertices of any triangle are the in- and escribed centres of the pedal triangle of the given triangle.

**11.** Given the pedal triangle DEF of a certain triangle ABC shew how to construct ABC.

**12.** If DEF is the pedal  $\triangle$  of the  $\triangle ABC$ , the triangles DEC, EFA, FDB, ABC are all equiangular, and their circumcircles are all of the same size.

**13.** In the figure of Art. 48 prove that  
(1) the triangles  $BI_1C$ ,  $CI_2A$ ,  $AI_3B$  are equiangular.  
(2) the four circles each of which passes through three of the four points I,  $I_1$ ,  $I_2$ ,  $I_3$  are all equal.

**14.** The internal and external bisectors of the angle A of a triangle meet the base BC in E, E' and the circumcircle in D and D'; prove that D is the orthocentre of the  $\triangle EE'D'$ .

**15.** O is the orthocentre of a triangle ABC, and D, E, F the circumcentres of the triangles OAB, OBC, and OCA. Prove that ABC and DEF are equal in all respects:

**16.** ABCD is a quadrilateral inscribed in a circle; E and F are the orthocentres of the  $\triangle^s$  ABC and ABD respectively; prove that CDFE is a  $\parallel^m$ .

Shew also that the orthocentres of the four  $\triangle^s$  ABC, BCD, CDA, DAB form a quadrilateral equal and similar to the given one.

[Use the Corollary to Art. 8 in this and the three following Exercises.]

**17.** Prove that the sum of the squares on the straight line joining the vertex of a triangle to the orthocentre, and on the opposite side = the sq. on the diameter of the circumcircle.

**18.** The orthocentre of a triangle is joined to the middle point of a side; prove that this joining line produced will meet the circumcircle in a point which is at the other end of the diameter through the angular point opposite to the bisected side.

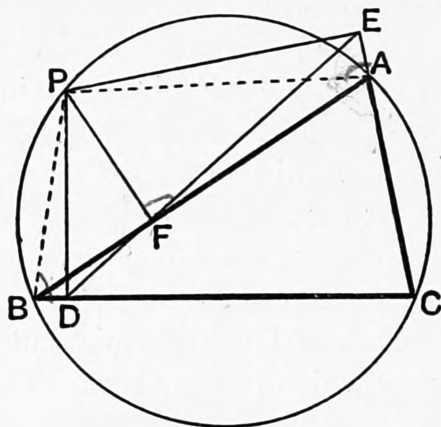
**19.** A line MN of given length slides between two fixed lines OM, ON; prove that the orthocentre of the  $\triangle OMN$  always lies on a certain circle.

**20.** A circular piece of paper is folded into any inscribed triangle; prove that the arcs of the folded segments all pass through a fixed point.



49. If from any point in the circumference of the circle described round a triangle perpendiculars be drawn to the sides of the triangle, the three points of intersection are in the same straight line.

Let  $ABC$  be a triangle,  $P$  any point on the circumcircle; from  $P$  draw  $PD$ ,  $PE$ ,  $PF$  perpendiculars to the sides  $BC$ ,  $CA$ ,  $AB$  respectively:  $D$ ,  $E$ ,  $F$  shall be in the same straight line.



[We will suppose that  $P$  is on the arc cut off by  $AB$ , on the opposite side from  $C$ , and that  $E$  is on  $CA$  produced through  $A$ ; the demonstration will only have to be slightly modified for any other figure.]

A circle will go round  $PEAF$ ; [III. 22 converse.

$$\therefore \angle PFE = \angle PAE \quad \text{[III. 21.]}$$

$$= \text{two rt. } \angle^s - \angle PAC \quad \text{[I. 13.]}$$

$$= \angle PBC. \quad \text{[III. 22.]}$$

Again, a circle will go round  $PFDB$ ; [III. 21.]

$$\therefore \angle PFD = \text{two rt. } \angle^s - \angle PBC. \quad \text{[III. 22 converse.]}$$

$$\therefore \angle^s PFD \text{ and } PFE \text{ are together} = \text{two right } \angle^s.$$

$\therefore EF$  and  $FD$  are in the same straight line.

*Conversely.* If the feet of the perpendiculars  $PD$ ,  $PE$ ,  $PF$  drawn from any point  $P$  to the sides of a triangle  $ABC$  are collinear, then  $P$  lies on the circumcircle of  $ABC$ .

For  $\angle BPD = \angle BFD$  (III. 21)  $= \angle AFE = \angle APE$  (III. 21).

$$\therefore \angle PBD = \text{rt. } \angle - \angle BPD = \text{rt. } \angle - \angle APE = \angle PAE.$$

$$\therefore \angle PBD + \angle PAC = 2 \text{ rt. } \angle^s.$$

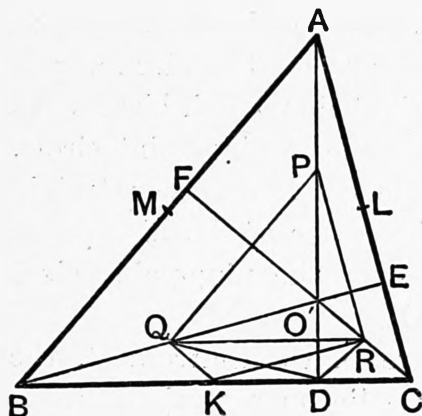
$\therefore P$ ,  $A$ ,  $B$ ,  $C$  lie on a circle.

**Definition.** The above straight line  $DFE$  is sometimes called the **Pedal Line** of  $P$ , and sometimes it is called the **Simson Line** of  $P$  from the name of its supposed discoverer.

50.  $ABC$  is a triangle, and  $O'$  is its orthocentre; the circle which passes through the middle points of  $O'A$ ,  $O'B$ ,  $O'C$  will pass through the feet of

the perpendiculars and through the middle points of the sides of the triangle.

Let  $P, Q, R$  be the middle points of  $OA, OB, OC$  respectively; let  $D$  be the foot of the perpendicular from  $A$  on  $BC$ , and  $K$  the middle point of  $BC$ ,



Then  $O'DB$  is a right-angled triangle and  $Q$  is the middle point of the hypotenuse  $O'B$ ;

$$\therefore QD = QO';$$

$$\therefore \angle QDO' = \angle QO'D.$$

Similarly, the  $\angle RDO' = \angle RO'D$ ;

$$\therefore \angle RDQ = \angle RO'Q.$$

But  $\angle^s CO'B, BAC$  together = two rt.  $\angle^s$ .

[III. 22.]

$\therefore \angle^s RDQ, BAC$  together = two rt.  $\angle^s$ .

Also  $\angle BAC = \angle QPR$ ,

[Art. 1.]

since  $QP, PR$  are  $\parallel$  to  $BA, AC$ .

$\therefore \angle^s RDQ, QPR$  = two rt.  $\angle^s$ .

$\therefore D$  is on the circle through  $P, Q, R$ .

Again,  $RK$  is parallel to  $O'B$ , and  $QK$  parallel to  $O'C$ ; [Art. 1.]

$$\therefore \angle QKR = \angle QO'C = \angle RDQ;$$

$\therefore K$  is also on the circumference of the circle.

Similarly, the two points in each of the other sides of the triangle  $ABC$  may be shewn to be on the circle.

**51.** The circle which is thus shewn to pass through these nine points is called the *Nine-point circle*: It has some curious properties, some of which we will now give.

**52.** The radius of the nine-point circle is half of the radius of the circle described about the original triangle.

For the  $\triangle PQR$  has its sides respectively halves of the sides of the triangle  $ABC$ , so that the triangles are equiangular. Hence the radius

of the circle described round PQR is half of the radius of the circle described round ABC.

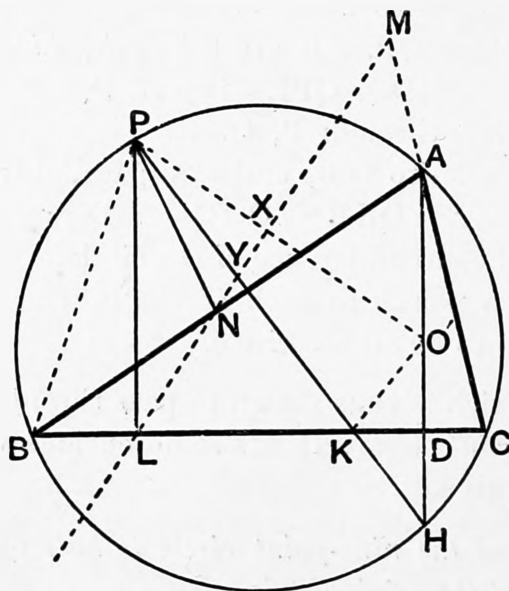
**53.** If  $O$  be the centre of the circle described round the triangle  $ABC$ , the centre of the nine-point circle is the middle point of  $OO'$ .

For  $OK$  is at right angles to  $BC$ , and therefore parallel to  $O'D$ . Hence the straight line which bisects  $KD$  at right angles must bisect  $OO'$ . And  $K$  and  $D$  are on the circumference of the nine-points circle, so that the straight line which bisects  $KD$  at right angles must pass through the centre of the nine-point circle. Similarly, from the other sides of the triangle  $ABC$  two other straight lines can be obtained, which pass through the centre of the nine-point circle and also bisect  $OO'$ . Hence the centre of the nine-point circle must coincide with the middle point of  $OO'$ .

**Cor.** The circumcentre  $O$ , the centroid  $G$ , the nine-point centre  $N$ , and the orthocentre  $O'$  lie on a straight line; also  $G$  is a point of trisection [Art. 7], and  $N$  the point of bisection of  $OO'$ .

**54.** We may state that the nine-point circle of any triangle touches the inscribed circle and the escribed circles of the triangle: a demonstration of this theorem will be found in Dr. Todhunter's *Plane Trigonometry*, Chapter XXIV. For the history of the theorem see the *Nouvelles Annales de Mathématiques* for 1863, page 562.

**55.** The line joining any point  $P$  on the circumcircle of any triangle  $ABC$  to the orthocentre  $O'$  is bisected by the pedal line of  $P$  with respect to the triangle.



Draw  $PL$ ,  $PM$ ,  $PN$  perpendicular to the sides of  $ABC$ ; then  $LMN$  is the pedal line of  $P$ .

Draw AD perpendicular to BC and produce it to meet the circum-circle in H. Then, if  $O'$  be the orthocentre, it lies on AD and  $O'D = DH$ . [Ex. 2, Art. 48.]

Let  $O'P$  meet LM in X. Then we have to prove that  $O'X = XP$ .

Join HP and let it meet BC in K. Join  $O'K$ .

Since PNB, PLB are both right angles, a circle goes through P, N, L, B;

$$\begin{aligned} \therefore \angle PLN &= \angle PBN = \angle PBA = \angle PHA & [\text{III. 21.}] \\ &= \angle HPL, \text{ since AH, PL are parallel.} \end{aligned}$$

$$\therefore \angle YLP = \angle YPL.$$

$$\therefore YP = YL,$$

and thus Y is the centre of the circumcircle of the  $\triangle PLK$ .

$$\therefore PY = YK = YL.$$

Again, since  $O'D = DH$ , and  $O'DK$ ,  $HDK$  are right  $\angle^s$ ,

$$\begin{aligned} \therefore \angle O'KD &= \angle HKD & [\text{I. 4.}] \\ &= \angle YKL = \angle YLK, \text{ since } YK = YL. \text{ (Proved.)} \end{aligned}$$

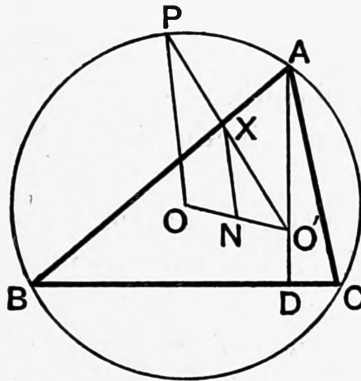
$\therefore O'K$  and LNM are parallel.

$\therefore$  LNM is a straight line drawn through the middle point Y of the side PK of the  $\triangle PKO'$  parallel to its base  $KO'$ .

$\therefore$  LNM bisects  $PO'$ , i.e.  $PX = XO'$ .

[Art. 1.]

**56.** *The middle point of the straight line joining any point P on the circumcircle of any triangle to the orthocentre  $O'$  lies on the nine-point circle of the triangle.*



Let O be the centre of the circumcircle. Bisect  $OO'$  in N and join NX.

Then, by Art. 53, N is the centre of the nine-point circle.

Hence, since X and N are the middle points of  $O'P$  and  $O'O$  respectively, NX is parallel to, and is one-half of, OP.

$$\therefore NX = \text{one-half of the circum-radius}$$

$$= \text{the radius of the nine-point circle.}$$

[Art. 52.]

$\therefore$  X lies on the nine-point circle.

Hence the intersections of  $O'P$  with the pedal line of P always lies on the nine-point circle.

## EXERCISES.

1. In the figure of Art. 50 prove that PQKL, PMKR are rectangles, and hence that the circle on PK as diameter goes through Q, L, M, R and also through D, E, F. [This is an easy method of proof of Art. 50.] Prove that PK, QL, RM meet at the centre of the nine-point circle.

2. If the perpendicular from any point P on the side BC of a  $\triangle ABC$  meet the circumcircle in Y, then AY is  $\parallel$  to the pedal line of P.

3. If through any point O on a circle three chords be drawn, and on each a circle be described, shew that these three circles, besides intersecting in O, meet in three points lying on a straight line.

[The three points are the feet of the  $\perp^{\text{rs}}$  from O upon the sides of the  $\triangle$  whose angular points are the other ends of the chords through O.]

4. *The pedal lines of a triangle with respect to two points on the circumcircle is equal to half the angle the two points subtend at the circumcentre.*

For, in the figure of Art. 55,

$$\angle NLC = \angle LNB + B = \angle LPB + B = \text{rt. } \angle - \angle PBL + B$$

$$= \text{rt. } \angle - \angle PBN = \text{rt. } \angle - \frac{1}{2} \angle POA, \text{ where O is the circumcentre,}$$

that is, inclination of pedal line of P to BC

$$= \text{rt. } \angle - \frac{1}{2} \angle POA.$$

So the inclination of the pedal line of any other point P' to BC

$$= \text{rt. } \angle - \frac{1}{2} \angle P'OA.$$

$\therefore$  angle between these pedal lines

$$= \text{the difference of these inclinations} = \frac{1}{2} \angle P'OA - \frac{1}{2} \angle POA$$

$$= \frac{1}{2} \angle P'OP.$$

If P, P' are at the extremities of a diameter of the circumcircle, then  $\frac{1}{2} \angle P'OP = \text{a right } \angle$ , and  $\therefore$  the corresponding pedal lines are perpendicular.

5. *The pedal lines of the ends of a diameter of the circumcircle meet at right angles on the nine-point circle.*

For if the ends of the diameter be P, P', and X, X' be the middle points of O'P, O'P', then XX' bisects OO' and thus passes through the nine-point centre. XX' is thus a diameter of the nine-point circle. Also the pedal lines of P, P' pass through X, X', and they meet at a right  $\angle$ , by the last example;  $\therefore$  they meet on the nine-point circle.

[III. 31.]

6. P is any point on the circumcircle of the triangle ABC, and PL, drawn parallel to BC, meets the circle in Q; prove that AQ is perpendicular to the pedal line of P.

**7.**  $I$  is the in-centre and  $I_1, I_2, I_3$  are the e-centres of a triangle  $ABC$ ; prove that the straight lines  $II_1, II_2, II_3, I_2I_3, I_3I_1, I_1I_2$  are all bisected by the circumcircle of the triangle  $ABC$ .

Prove also that the middle points of  $II_1$  and  $I_2I_3$  are at the ends of a diameter of this circle.

**8.**  $A, A', B, C$  are four points on a circle. Prove that the pedal line of  $A'$  with respect to the  $\triangle ABC$ , of  $A$  with respect to the  $\triangle A'BC$ , and the nine-point circles of these two  $\triangle$ 's meet in a point.

[This point is at the intersection of the diagonals of the parallelogram formed by  $A, A'$  and the orthocentres of the two triangles.]

**9.** The nine-point circles of the triangles formed by four points, taken three at a time, meet in a point.

[Let  $A, B, C, D$  be the points, and  $P, Q, R, S, T, U$  the middle points of  $BC, CA, AB, BD, CD, AD$ . If  $O$  be the intersection of the nine-point circles of  $ABC$  and  $BCD$ , then  $\angle POQ = \angle PRQ = \angle ACB$ , and  $\angle POT = \angle PST = \angle BCD$ ;  $\therefore \angle QOT = \angle ACD = \angle QUT$ ;  $\therefore$  etc.]

**10.** If  $A, B, C, D$  be four points on a circle, the pedal lines of the triangles  $ABC, BCD, CDA, DAB$  with respect to  $D, A, B, C$  respectively meet in a point. [Use the two previous Exercises.]

**11.** Prove also that the centroids of the four triangles are concyclic.

**12.** A triangle inscribed in a given circle has its orthocentre at a fixed point. Prove that the middle points of its sides lie on a fixed circle.

**13.** Having given an angular point  $A$  of a triangle, its circumcentre, and the length of the base  $BC$ , the loci of its orthocentre and its nine-point centre are both circles.

**14.** Four concyclic points, taken three by three, determine four triangles whose nine-point centres are concyclic.

**15.** If the nine-point circle and one of the angular points of a triangle be given, the locus of the orthocentre is a circle.

**16.**  $O$  is the orthocentre of a triangle  $ABC$ ; the nine-point circles of the triangles  $OAB, OBC, OCA, ABC$  all coincide.

**17.**  $D$  and  $E$  are points on the circumcircle of the triangle  $ABC$ , and their pedal lines meet in  $P$ ; prove that the locus of  $P$  is a circle when  $A$  moves on the circumcircle and  $B, C, D, E$  are fixed.

## THE TANGENCIES OF CIRCLES.

**57.** A circle can be found to satisfy three conditions in general. Conversely, if three independent conditions be given, one circle, or at any rate only a finite number of circles can be found to satisfy them.

*To describe a circle which shall pass through three given points not in the same straight line.*

This is solved in Euclid IV. 5.

**58.** *To describe a circle which shall pass through two given points on the same side of a given straight line, and touch that straight line.*

Let  $A$  and  $B$  be the given points ; join  $AB$  and produce it to meet the given straight line at  $C$ . Make a square equal to the rectangle  $CA$ ,  $CB$  (II. 14), and on the given straight line take  $CE$  equal to a side of this square.

Describe a circle through  $A$ ,  $B$ ,  $E$  (IV. 5); this will be the circle required (III. 37).

Since  $E$  can be taken on either side of  $C$ , there are two solutions.

The construction fails if  $AB$  is parallel to the given straight line. In this case bisect  $AB$  at  $D$ , and draw  $DC$  at right angles to  $AB$ , meeting the given straight line at  $C$ . Then describe a circle through  $A$ ,  $B$ ,  $C$ .

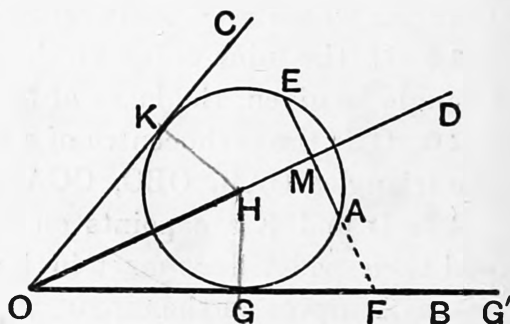
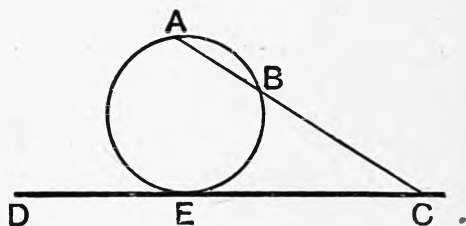
**59.** *To describe a circle which shall pass through a given point and touch two given straight lines.*

Let  $A$  be the given point, and  $OB$ ,  $OC$  the given straight lines. Bisect the angle  $BOC$  by the straight line  $OD$ . Draw  $AM$  perpendicular to  $OD$  and produce it to  $E$ , so that  $AM = ME$ .

Through  $A$ ,  $E$ , draw a circle (by Art. 58) to touch the given straight line  $OB$ . This is done by producing  $EA$  to meet  $OB$  in  $F$  and taking  $FG^2 = FA \cdot FE$ .

Since the circle passes through  $A$  and  $E$ , its centre lies on  $MO$  which bisects  $AE$  at right  $\angle^s$ . [III. 1.

Hence, if  $H$  be the centre, since it lies on the straight line  $OD$  which bisects the  $\angle BOC$  the perpendiculars  $HG$ ,  $HK$  on  $OB$ ,  $OC$  are equal,



and therefore the circle just drawn also touches OC. It is therefore the required circle.

There are two such circles ; for if we take  $G'$  on the other side of  $F$  from  $G$ , such that  $FG' = FG$ , a second circle can be drawn through  $A$ ,  $E$ , and  $G'$  which will also satisfy the required conditions.

If  $A$  is on one of the given straight lines, draw from  $A$  a straight line at right angles to this given straight line ; the point of intersection of this straight line with either of the two straight lines which bisect the angles made by the given straight lines may be taken for the centre of the required circle.

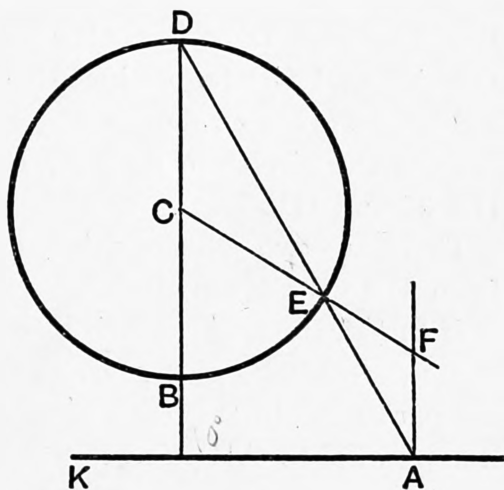
If the two given straight lines are parallel, instead of drawing a straight line  $BC$  to bisect the angle between them, we must draw it parallel to them, and equidistant from them.

**60.** *To describe a circle which shall touch three given straight lines, not more than two of which are parallel.*

Proceed as in Euclid IV. 4. If the given straight lines form a triangle, four circles can be described, namely, one as in Euclid, and three others, as on Page 181, each touching one side of the triangle and the other two sides produced. If two of the given straight lines are parallel, two circles can be described, namely, one on each side of the third given straight line.

**61.** *To describe a circle which shall touch a given circle, and touch a given straight line at a given point.*

Let  $A$  be the given point in the given straight line  $AK$ , and  $C$  be the centre of the given circle. Through  $C$  draw a straight line perpendicular to the given straight line to meet the circle in  $B$  and  $D$ , of which  $D$  is the more remote from the given straight line. Join  $AD$ , meeting the circle in  $E$ . From  $A$  draw a straight line at right angles to the given straight line, meeting  $CE$  produced at  $F$ .



*Then  $F$  shall be the centre of the required circle, and  $FA$  its radius.*

For the  $\angle AEF = \text{the } \angle CED$  ; [I. 15.]

and the  $\angle EAF = \text{the } \angle CDE$  ; [I. 29.]

$\therefore \text{the } \angle AEF = \text{the } \angle EAF$  ;

$\therefore AF = EF$ .

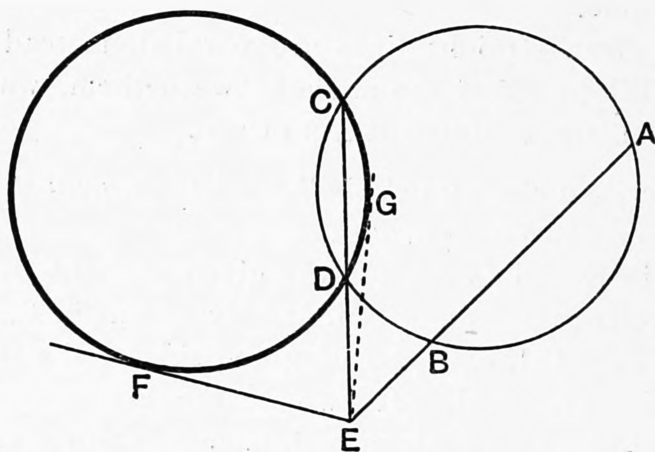
[I. 6.]



In a similar manner another solution may be obtained by joining  $AB$ . If the given straight line falls without the given circle, the circle obtained by the first solution touches the given circle externally, and the circle obtained by the second solution touches the given circle internally.

If the given straight line cuts the given circle, both the circles obtained touch the given circle externally.

✓ **62.** *To describe a circle which shall pass through two given points and touch a given circle.*



Let  $A$  and  $B$  be the given points. Take any point  $C$  on the circumference of the given circle, and describe a circle through  $A$ ,  $B$ ,  $C$ . If this described circle touches the given circle, it is the required circle. But if not, let  $D$  be the other point of intersection of the two circles. Let  $AB$  and  $CD$  be produced to meet at  $E$ ; from  $E$  draw a tangent  $EF$  to the given circle.

Then the circle through  $A$ ,  $B$ ,  $F$  shall be the required circle. See III. 35 and III. 37.

There are two solutions, because two tangents  $EF$ ,  $EG$  can be drawn from  $E$  to the given circle.

If the straight line which bisects  $AB$  at right angles passes through the centre of the given circle, the construction fails, for  $AB$  and  $CD$  are parallel. In this case  $F$  must be determined by drawing a straight line parallel to  $AB$  so as to touch the given circle.

**63.** *To describe a circle which shall touch two given straight lines and a given circle.*

Draw two straight lines parallel to the given straight lines, at a distance from them equal to the radius of the given circle, and on the sides of them remote from the centre of the given circle. Describe a

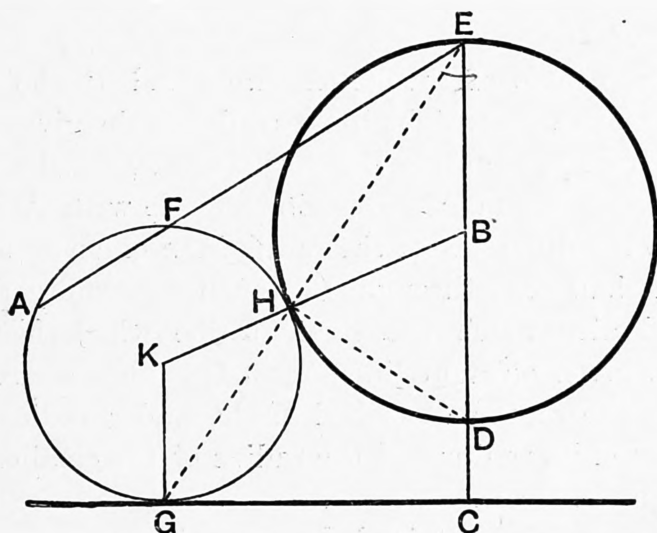
circle touching the straight lines thus drawn, and passing through the centre of the given circle (Art. 59). A circle having the same centre as the circle thus described, and a radius equal to the excess of its radius over that of the given circle, will be the required circle.

Two solutions will be obtained, because there are two solutions of the problem in Art. 59; the circles thus obtained touch the given circle externally.

We may obtain two circles which touch the given circle internally, by drawing the straight lines parallel to the given straight lines on the same sides of them as the centre of the given circle.

✓ **64.** *To describe a circle which shall pass through a given point and touch a given straight line and a given circle.*

We will suppose the given point and the given straight line without the circle; other cases of the problem may be treated in a similar manner.



Let A be the given point, and B the centre of the given circle. From B draw a perpendicular to the given straight line, meeting it at C, and meeting the circumference of the given circle at D and E, so that D is between B and C.

Join EA and determine a point F in EA, produced if necessary, such that the rect. EA, EF = the rect. EC, ED; this can be done by describing a circle through A, C, D, which will meet EA at the required point.

[III. 36, Corollary.

Describe a circle to pass through A and F and touch the given straight line (Art. 58); this shall be the required circle.

For, let the circle thus described touch the given straight line at G ; join EG meeting the given circle at H, and join DH.

Since the  $\angle^s$  GHD, GCD are right  $\angle^s$ , [III. 31, and *Construction*.

$\therefore$  H, G, C, D lie on a circle.

$\therefore$  rect. EC, ED = rect. EG, EH. [III. 36.

$\therefore$  H is on the described circle. [III. 36, *Corollary*.

Take K the centre of the described circle ; join KG, KH, and BH. Then  $\angle KHG = \angle KGH$  [I. 5] =  $\angle HEB$  [I. 29] =  $\angle EHB$  [I. 5].

$\therefore$  KHB is a straight line ;

$\therefore$  the described circle touches the given circle.

Two solutions will be obtained, because there are two solutions of the problem in Art. 58 ; the circles thus described touch the given circle externally.

By joining DA instead of EA we can obtain two solutions in which the circles described touch the given circle internally.

✓ **65.** *To describe a circle which shall touch a given straight line and two given circles.*

Let A be the centre of the larger circle and B the centre of the smaller circle. Draw a straight line parallel to the given straight line, at a distance from it equal to the radius of the smaller circle, and on the side of it remote from A. Describe a circle with A as centre, and radius equal to the difference of the radii of the given circles. Describe a circle which shall pass through B, touch externally the circle just described, and also touch the straight line which has been drawn parallel to the given straight line [Art. 64]. Then a circle having the same centre as the second described circle, and a radius equal to the difference between its radius and the radius of the smaller given circle, will be the required circle.

Two solutions will be obtained, because there are two solutions of the problem in Art. 64 ; the circles thus described touch the given circles externally.

We may obtain in a similar manner circles which touch the given circles internally, and also circles which touch one of the given circles internally and the other externally.

[For other cases of circles satisfying given conditions see Arts. 87, 88, 90, and 91.]

## CONSTRUCTIONS.

**66.** Construct a triangle given its base, its vertical angle, and the sum of its sides equal to a given straight line.

Let  $AB$  be the base; on it describe a segment of a circle containing an  $\angle ACB$  equal to the given vertical angle. Let  $C$  be a point on such that the sum of  $AC$ ,  $CD$  is equal to the given sum.

Produce AC to D so that  $CD=CB$ ,  
and join DB.

Then AD is equal to the given straight line.

Also the  $\angle ACB$  = the sum of the  $\angle^s$  CDB and CBD [I. 32], that is,  
= twice the  $\angle CDB$  [I. 5].

$\therefore$  the  $\angle$  ADB is half of the given vertical  $\angle$ .

Hence we have the following solution. Describe on AB a segment of a circle containing an  $\angle$  equal to the given  $\angle$  and a second segment containing an  $\angle$  equal to half the given  $\angle$ .

With A as centre, and a radius equal to the given straight line, describe a circle.

Join A with a point of intersection D of this circle and the second segment; this joining straight line will cut the circumference of the first segment at a point C which solves the problem.

The given straight line must exceed AB, and it must not exceed a certain straight line which we will now determine. Suppose the circumference of the first segment bisected at E; join AE, and produce it to meet the circumference of the second segment at F.

Then  $AE = EB$  [III. 29], and  $EB = EF$ ,

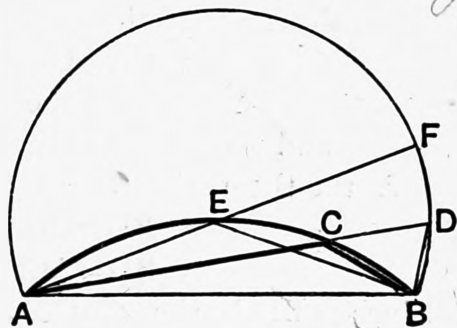
since  $\angle^s$  EFB,  $EBF = \angle AEB = \angle ACB$

$$= \text{twice } \angle ADB = \text{twice } \angle AFB,$$

and  $\therefore \angle \text{EBF} = \angle \text{EFB}$ .

Thus EA, EB, EF are all equal ;

∴ E is the centre of the circle of which ADB is a segment. [III. 9.  
Hence AF is the longest straight line which can be drawn from A to  
the circumference of the described segment ; so that the given straight  
line must not exceed twice AE.



**67.** *To describe an isosceles triangle having each of the angles at the base double of the third angle.*

This problem is solved in IV. 10; we may suppose the solution to have been discovered by such an analysis as the following.

Suppose the triangle ABD such a  $\triangle$  as is required, so that each of the  $\angle^s$  at B and D is double of the  $\angle$  at A.

Bisect the  $\angle$  at D by the straight line DC.

Then the  $\angle ADC =$  the  $\angle$  at A;

$$\therefore CA = CD.$$

The  $\angle CBD =$  the  $\angle ADB$ , by hypothesis,

and the  $\angle CDB =$  the  $\angle$  at A;

$\therefore$  the third  $\angle BCD =$  the third  $\angle ABD$ . [I. 32.]

$$\therefore BD = CD. \quad [I. 6.]$$

$$\therefore BD = AC.$$

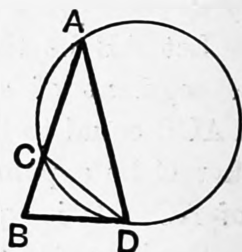
Since  $\angle BDC =$  the  $\angle$  at A, BD is a tangent at D to the circle described round the  $\triangle ACD$  (Note on III. 32).

$\therefore$  the rect. AB, BC = the square on BD. [III. 36.]

$\therefore$  the rect. AB, BC = the square on AC.

$\therefore$  AB is divided at C in the manner required in II. 11.

Hence the synthetical solution of the problem is evident.



## EXERCISES.

Describe a triangle having given its base, its vertical angle, and

1. another side;
2. its altitude;
3. the point where the perpendicular from the vertical angle meets the base;

4. the difference of its sides;

5. the sum of the squares on its sides;

[By Page 145 the vertex lies on a known arc of a circle going through the ends of the base; also, by Page 109, Ex. 1, the vertex lies on a circle whose centre is the middle point of the base; the vertex is thus at the intersection of these two circles.]

6. the difference of the squares on its sides;

7. the point where the bisector of the vertical angle meets the base;

8. the length of the median through the vertex;

9. the length of the median through one end of the base;

10. the difference of the angles at the ends of its base.

Describe a circle

11. with a given radius to touch two given straight lines;

12. with a given radius to touch two given circles;

**13.** with a given radius to touch a given circle and a given straight line ;

[Since the required circle is to touch the given circle its centre is somewhere on a circle whose centre is that of the given circle and whose radius = sum of radius of given circle and given radius ; since it touches given straight line the centre is on a straight line parallel to the given one ;  $\therefore$  etc.]

**14.** with a given radius to pass through a given point and touch a given straight line ;

**15.** with a given radius and centre in a given straight line to touch another given straight line ;

**16.** through a given point to touch a given straight line at a given point ;

**17.** to touch a given circle at a given point and also to touch a given straight line ;

**18.** through a given point to touch a given circle at a given point ;

**19.** to touch a given circle and also to touch two tangents to the circle ;

**20.** to touch a given straight line at a given point and bisect the circumference of a given circle ;

**21.** to pass through a given point and bisect the circumferences of two given circles ;

**22.** Inscribe a circle in a given sector of a circle.

**23.** Find the centre of a circle cutting off equal chords from the sides.

**24.** With three given points as centre draw three circles which touch in pairs.

**25.** Find a point outside a circle such that the tangents from it to a circle meet at a given angle.

**26.** Draw a chord in a given circle equal to one given straight line and parallel to another.

Describe a triangle having given

**27.** the centres of its three escribed circles ;

**28.** its in-centre and the centres of two escribed circles ;

**29.** its pedal triangle ;

**30.** its base, altitude, and circum-radius ;

**31.** an  $\angle A$ , the perp<sup>r</sup> from A upon the opposite side, and the in-radius ;

**32.** the vertical  $\angle$ , the perimeter, and the in-radius ;

**33.** the vertical  $\angle$ , the perimeter, and the altitude ;

**34.** the base, one  $\angle$  at the base, and the in-radius.

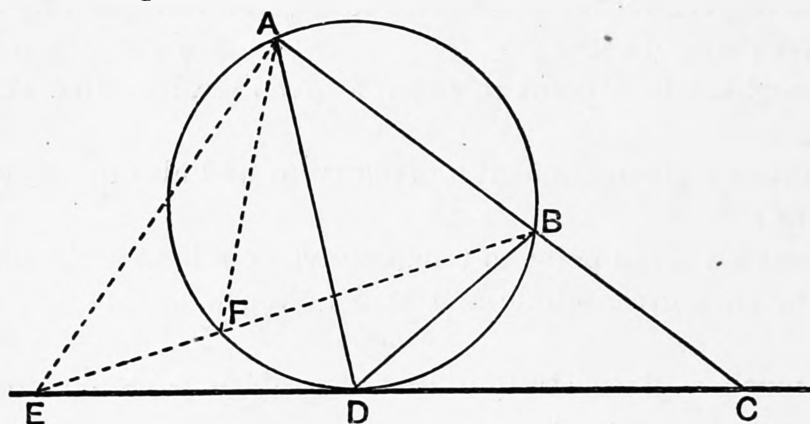
**35.** an angular point, the ortho-centre, and the circum-centre.

MAXIMA AND MINIMA [BOOKS III. AND IV.].

**68.** A and B are two given points on the same side of a given straight line, and AB produced meets the given line at C; of all points in the given straight line on each side of C, it is required to determine that at which AB subtends the greatest angle.

Describe a circle to pass through A and B, and to touch the given straight line on that side of C which is to be considered. Let D be the point of contact ;

D shall be the required point.



For, take any other point E in CD on the same side of C as D; draw EA, EB; then one at least of these straight lines will cut the circumference ADB.

Suppose that BE cuts the circumference at F; join AF. Then the  $\angle AFB =$  the  $\angle ADB$  [III. 21]; and the  $\angle AFB >$  the  $\angle AEB$  [I. 16];

$\therefore$  the  $\angle ADB >$  the  $\angle AEB$ .

$\therefore$  the  $\angle$  subtended by A, B at D is greater than the  $\angle$  subtended at any other point of CD.

**69.** A and B are two given points within a circle ; and AB is drawn and produced both ways so as to divide the whole circumference into two arcs ; it is required to determine the point in each of these arcs at which AB subtends the greatest angle.

Describe a circle to pass through A and B and to touch the circumference considered [Art. 62]; the point of contact will be the required point. The demonstration is similar to that in the preceding proposition.

**70.** A and B are two given points without a given circle; it is required to determine the points on the circumference of the given circle at which AB subtends the greatest and least angles.

Suppose that neither  $\overline{AB}$  nor  $\overline{AB}$  produced cuts the given circle.

Describe two circles to pass through A and B, and to touch the



given circle [Art. 62]; then G the point of contact of the circle which touches the given circle externally, will be the point where the angle is greatest, and F, the point of contact of the circle which touches the given circle internally, will be the point where the angle is least. The demonstration is similar to that in Art. 68.

If AB cuts the given circle, both the circles obtained by Art. 62 touch the given circle internally; in this case the angle subtended by AB at a point of contact is less than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB. Here the angle is greatest at the points where AB cuts the circle, and is there equal to two right angles.

If AB *produced* cuts the given circle, both the circles obtained by Art. 62 touch the given circle externally; in this case the angle subtended by AB at a point of contact is greater than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB. Here the angle is least at the points where AB produced cuts the circle, and is there zero.

### EXERCISES.

**\*\*1.** Given the base and vertical angle, construct it when its area is greatest, and show that the tangent at its vertex is then parallel to its base.

**\*\*2.** Of all triangles inscribed in a circle, the equilateral one is the greatest.

[For if any triangle is such that the tangent at one vertex A is not  $\parallel$  to the base BC, we can by Ex. 1 obtain a greater  $\triangle$  by keeping BC fixed, and taking the vertex at a point where the tangent is  $\parallel$  to the base;  $\therefore$  the greatest  $\triangle$  has the tangent at each angular point  $\parallel$  to the opposite side;  $\therefore$  etc.]

**3.** The pedal triangle of a triangle ABC is the triangle of least perimeter with its vertices on the sides of ABC. [Use Art. 27.]

**4.** Of all quadrilaterals inscribed in a circle, the inscribed square has the greatest area and the greatest perimeter.

**5.** Of all chords drawn through a given point within a circle, the least is the one that is bisected at the given point; prove also that it cuts off the least area from the circle.

**6.** Given a circle and two tangents, prove that the tangent which is such that the portion of it intercepted between the two tangents is a minimum, is bisected at the point of contact. Prove also that it cuts off a maximum or a minimum triangle.



## BOOK VI.

**71. Radical Axis.** The radical axis of two circles is the locus of a point which moves so that the lengths of the tangents drawn from it to the two circles are equal.

**72.** A geometrical construction can be given for the radical axis of two circles.

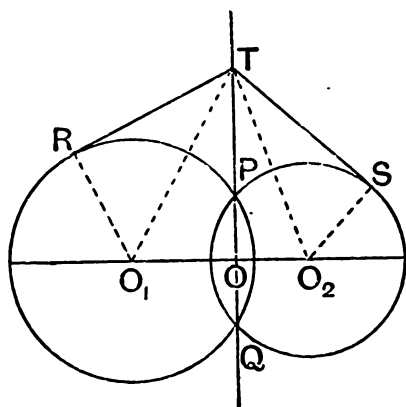


FIG. 1.

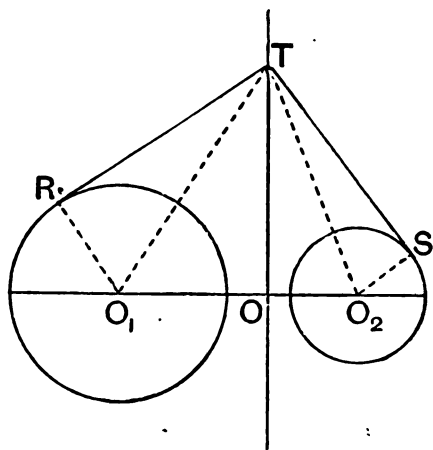


FIG. 2.

If the circles intersect in real points, P and Q, as in Fig. 1, the radical axis is clearly the straight line PQ. For if T be any point on PQ and TR and TS be the tangents from it to the circles, we have, by III. 36,

$$TR^2 = TP \cdot TQ = TS^2.$$

If they do not intersect in real points, as in the second figure, let T be a point such that the tangents TR and TS are equal in length.

Draw TO perpendicular to  $O_1O_2$ .

Since

$$TR^2 = TS^2,$$

$$\text{we have } TO_1^2 - O_1R^2 = TO_2^2 - O_2S^2,$$

$$\text{i.e. } TO^2 + O_1O^2 - O_1R^2 = TO^2 + OO_2^2 - O_2S^2,$$

$$\text{i.e. } O_1O^2 - OO_2^2 = O_1R^2 - O_2S^2,$$

$$\text{i.e. } (O_1O - OO_2)(O_1O + OO_2) = O_1R^2 - O_2S^2,$$

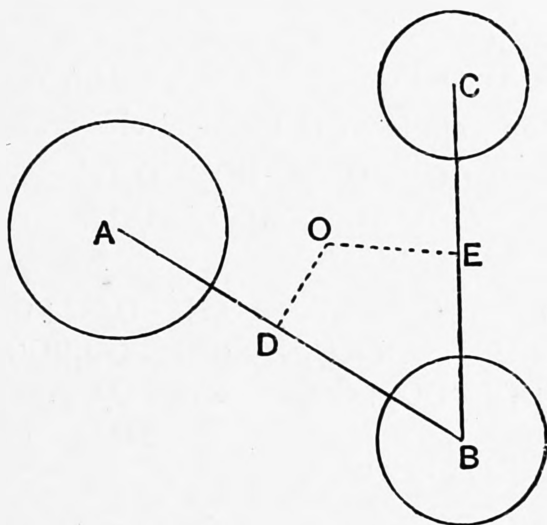
$$\text{i.e. } O_1O - OO_2 = \frac{O_1R^2 - O_2S^2}{O_1O_2} = \text{a constant quantity.}$$

Hence  $O$  is a fixed point since it divides the fixed straight line  $O_1O_2$  into parts whose difference is constant.

Therefore, since  $O_1OT$  is a right angle, the locus of  $T$ , *i.e.* the radical axis, is a straight line perpendicular to the line of centres.

**73.** If the radical axis be assumed to be a straight line it may also be constructed as follows : Draw any circle meeting one of the circles in  $E$ ,  $F$  and the other in  $G$ ,  $H$  ; let  $EF$  and  $GH$  meet in  $P$ , then by III. 36 rect.  $PE$ ,  $PF = \text{rect. } PG$ ,  $PH$ .  $\therefore$  tangents from  $P$  to the two circles are equal and  $P$  is on the radical axis. The straight line through  $P$  perpendicular to the line of centres will be the radical axis.

**74.** *The radical axis of three circles, taken in pairs, meet in a point, which is such that the tangents from it to the three circles are equal.*



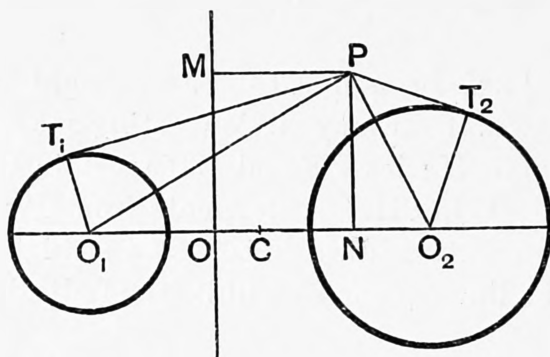
Let the three circles be called  $A$ ,  $B$ , and  $C$ , and let the radical axis of  $A$  and  $B$  and that of  $B$  and  $C$  meet in a point  $O$ .

By the definition of the radical axis the tangent from  $O$  to the circle  $A =$  the tangent from  $O$  to the circle  $B$ , and the tangent from  $O$  to the circle  $B =$  tangent from it to the circle  $C$ .

Hence the tangent from  $O$  to the circle  $A =$  the tangent from it to the circle  $C$ , *i.e.*  $O$  is also a point on the radical axis of the circles  $A$  and  $C$ .

This point  $O$  is called the **radical centre** of the three circles.

**75.** *The difference in the squares on the tangents drawn from any point P to two circles is equal to twice the rectangle contained by the distance between their centres and the perpendicular from P upon their radical axis.*



Let  $PT_1$ ,  $PT_2$  be the tangents from a point  $P$  to two circles;  $O_1$  and  $O_2$  their centres and  $OM$  their radical axis; draw  $PM$  and  $PN$  perpendicular to the radical axis and the line of centres. Let  $C$  be the middle point of  $O_1O_2$ .

$$\text{Then } PO_1^2 - PO_2^2 = O_1N^2 - O_2N^2.$$

Also, since the tangents from  $O$  to the circles are equal,

$$\begin{aligned} \therefore OO_2^2 - O_2T_2^2 &= OO_1^2 - O_1T_1^2, \\ O_2T_2^2 - O_1T_1^2 &= OO_2^2 - OO_1^2. \end{aligned}$$

$\therefore$  by addition,

$$\begin{aligned} PO_1^2 - O_1T_1^2 - (PO_2^2 - O_2T_2^2) &= O_1N^2 - O_2N^2 + OO_2^2 - OO_1^2 \\ &= (O_1N + O_2N)(O_1N - O_2N) + (OO_2 + OO_1)(OO_2 - OO_1) \\ &= O_1O_2(O_1N - OO_1 + OO_2 - O_2N) = 2 O_1O_2 \cdot ON \\ &= 2 O_1O_2 \cdot PM. \end{aligned}$$

Hence the proposition is true.

**Corollary.** If  $P$  lies on the circle of centre  $O_2$  then  $PT_2$  vanishes, and the theorem becomes: *The square of the tangent drawn from a point on one circle to another equals twice the rectangle contained by the perpendicular from the point on the radical axis of the circles and the distance between the centres of the circles.*

## EXERCISES.

**\*\*1.** The radical axis of two circles bisects either of their common tangents.

**\*\*2.** If, of three circles, each touches the other two, the common tangents at the points of contact are concurrent.

**\*\*3.** The locus of a point which moves so that the difference in the squares of the tangents drawn from it to two given circles is constant is a straight line parallel to the radical axis of the two circles.

**4.** Two circles meet in  $AB$ , and through  $A$  is drawn a chord  $PAQ$  cutting off from each circle segments containing equal angles. Prove that the tangents at  $P$  and  $Q$  meet in  $AB$  produced.

**5.** Prove the following construction for the radical axis of the circumcircle and the nine-point circle of a  $\triangle ABC$ : Let  $D, E, F$  be the feet of the perpendiculars on the sides  $BC, CA, AB$  of the  $\triangle$  from the opposite vertices; let  $EF$  and  $BC$  meet in  $L$ ,  $FD$  and  $CA$  in  $M$ , and  $DE$  and  $AB$  in  $N$ ; then  $LMN$  is the required radical axis.

**6.** The radical centre of the three circles on the sides of a triangle as diameters is the orthocentre of the triangle. [Use Page xxxvi, Ex. 6.]

**76. Coaxal Circles.** A system of circles is said to be coaxal when they have a common radical axis, *i.e.* when the radical axis of each pair of circles of the system is the same.

Since the radical axis of any pair of the circles is perpendicular to the line joining their centres, it follows that the centres of all such circles of a coaxal system must lie on a straight line which is perpendicular to the radical axis.

If two of the circles intersect in two points  $P$  and  $Q$ , as in Fig. 1,

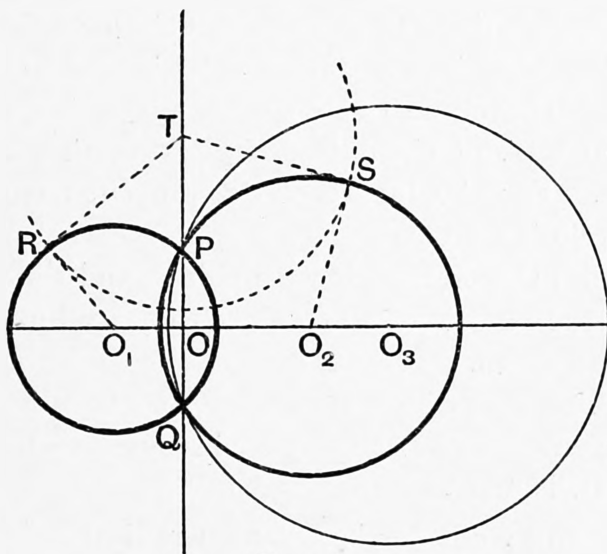


FIG. 1.

then all the circles of the system intersect in the same two points, and any circle of the system is then obtained by taking a point  $O_3$  on the line of centres as centre, and the length of the straight line joining this point to the point  $P$  or  $Q$  as radius. The circles are then said to be of the Intersecting Species.

When two of the circles do not intersect as in Fig. 2 then any circle of the system can be constructed as follows :

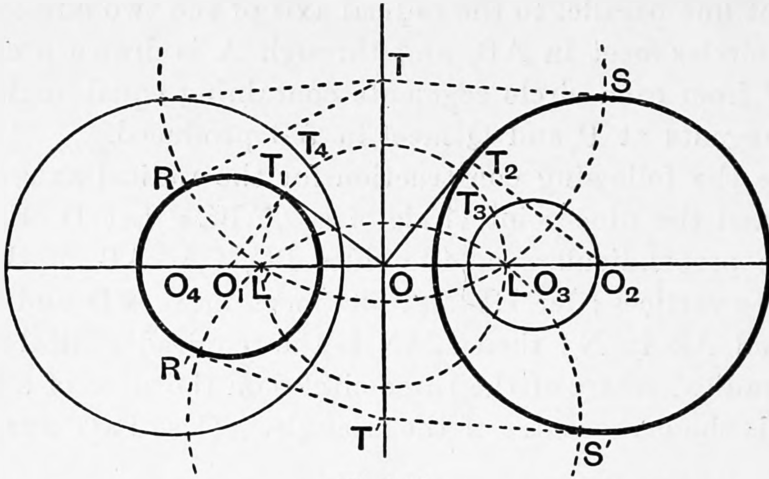


FIG. 2.

If  $O$  be the point of intersection of the radical axis and the line of centres, the length of the tangent drawn to every circle of the system from  $O$  is the same. If a circle be drawn with centre  $O$ , and radius equal to the tangent,  $OT_1$  or  $OT_2$ , drawn to the two given circles as radius, the points of contact of tangents from  $O$  to any circle of the system lie on this circle. To construct the circle of the system whose centre is *any* point  $O_3$ , we draw a tangent  $O_3T_3$  to the circle centre  $O$ ; with centre  $O_3$  and radius  $O_3T_3$  describe a circle; this circle is one of the system. For  $OT_3$  is the tangent to it from  $O$  and is of the required length. The circle (whose centre is either of the points of intersection,  $L$  and  $L'$ , of the dotted circle  $O$  with the line of centres) is of zero radius. For the length of the tangent from either  $L$  or  $L'$  to the dotted circle is zero.

These two circles therefore reduce to points, and the points  $L$  and  $L'$  are hence called the **Limiting Points**, since they are limiting positions of the circles of the system.

There are no circles of the system whose centres lie between  $O$  and  $L$ . For the tangent drawn from any such centre to the dotted circle, whose centre is  $O$ , is imaginary.

**77.** Let  $T$  be any point on the common radical axis of a system of coaxial circles and let  $TR$  be the tangent from it to any circle of the system.

Then a circle, whose centre is  $T$  and whose radius is  $TR$  (Fig. Art. 76) will cut each circle of the coaxial system orthogonally.

[For the radius  $TR$  of this circle is at right angles to the radius  $O_1R$ , and so for its intersection with any other circle of the system.]

Hence the limiting points (being point-circles of the system) are on this orthogonal circle.

The limiting points are therefore the intersections with the line of centres of *any* circle whose centre is on the common radical axis and whose radius is the tangent from it to any of the circles of the system.

Since, in Fig. 1, the limiting points are imaginary these orthogonal circles do not meet the line of centres in real points.

In Fig. 2 they pass through the limiting points  $L_1$  and  $L_2$ .

These orthogonal circles (since they all pass through two points, real or imaginary) are therefore a coaxal system.

Also if the original circles, as in Fig. 1, intersect in real points, the orthogonal circles intersect in imaginary points; in Fig. 2 the original circles intersect in imaginary points, and the orthogonal circles in real points.

We therefore have the following theorem :

*A set of coaxal circles can be cut orthogonally by another set of coaxal circles, the centres of each set lying on the radical axis of the other set; also one set is of the limiting-point species and the other set of the other species.*

**78.** Without reference to the limiting points of the original system, it may be easily found whether or not the orthogonal circles meet the original line of centres.

For the circle, whose centre is  $T$  and whose radius is  $TR$ , meets or does not meet the line  $O_1O_2$  according as  $TR^2$  is  $>$  or  $<$   $TO^2$ ,

*i.e.* according as  $TO_1^2 - O_1R^2$  is  $\geq TO^2$ ,

*i.e.* according as  $TO^2 + OO_1^2 - O_1R^2$  is  $\geq TO^2$ ,

*i.e.* according as  $OO_1$  is  $\leq O_1R$ ,

*i.e.* according as the radical axis is without, or within, each of the circles of the original system.

**79.** We can now deduce an easy construction for the circle that cuts any three circles orthogonally.

Consider the three circles in the figure of Art. 74.

By Art. 77 any circle cutting  $A$  and  $B$  orthogonally, has its centre on their common radical axis, *i.e.* on the straight line  $OD$ .

Similarly any circle cutting  $B$  and  $C$  orthogonally has its centre on the radical axis  $OE$ .

Any circle cutting all three circles orthogonally must therefore have its centre at the intersection of  $OD$  and  $OE$ , *i.e.* at the radical centre  $O$ . Also its radius must be the length of the tangent drawn from the radical centre to any one of the three circles.

## EXERCISES.

**\*\*1.** If there be three coaxial circles whose centres are  $O_1, O_2, O_3$ , the squares on the tangents drawn from any point of the third circle to the first two are to one another as  $O_1O_3$  to  $O_2O_3$ . [Use Art. 75, Cor.]

**\*\*2.** Conversely, if the squares on the tangents drawn from a point  $P$  to two circles (centres  $O_1$  and  $O_2$ ) be in a given ratio, the locus of  $P$  is a circle coaxial with the given ones, whose centre  $O_3$  is such that  $O_1O_3 : O_2O_3$  in the given ratio.

**\*\*3.** Any circle passing through the limiting points of a set of coaxial circles cuts them all orthogonally.

**4.** The radical axis of a circle centre  $O_1$  and a point  $O$  (considered as a point-circle) bisects at right angles the distance between  $O$  and the point in which its polar with respect to the circle meets  $OO_1$ .

**5.** Construct the circle which passes through a given point and cuts two circles orthogonally. [Treat the point as a point-circle, and apply Art. 79 and Ex. 4.]

**\*\*6.** The polar of either limiting point with respect to any circle of the system passes through the other.

[The polar of  $L$  with respect to the circle  $O_1$  goes through  $L'$  if

$$O_1T_1^2 = O_1L' \cdot O_1L = O_1O^2 - OL^2$$

$$\text{i.e. if } OO_1^2 = O_1T_1^2 + OL^2$$

$$= O_1T_1^2 + OT_1^2, \text{ which is true.}]$$

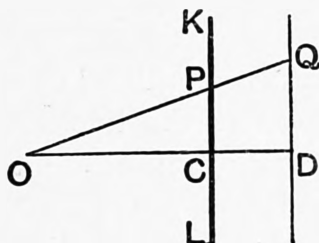
## EXAMPLES OF LOCI. [BOOK VI.]

**80.** *O is a fixed point from which any straight line is drawn meeting a fixed straight line KL at P; in OP a point Q is taken such that OQ is to OP in a fixed ratio: determine the locus of Q.*

We shall shew that the locus of Q is a straight line.

Draw  $OC \perp$  to KL and on it, produced if necessary, take a point D such that OD is to OC in the given ratio.

Join O to any point P on KL and on it take a point Q such that OQ is to OP in the fixed ratio; join QD.



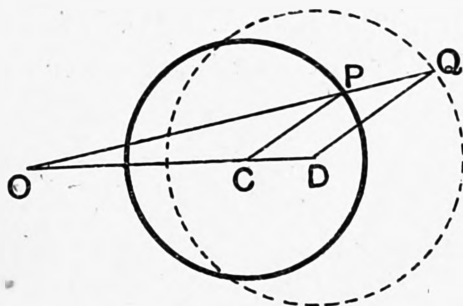
The  $\triangle$  ODQ and OCP are similar (VI. 6);

$\therefore$  the  $\angle ODQ =$  the  $\angle OCP$ , and  $\therefore =$  a right angle.

$\therefore$  Q lies in the straight line drawn through D at right  $\angle$ s to OD.

**81.** *O is a fixed point from which any straight line is drawn meeting a fixed circle at P; in OP a point Q is taken such that OQ is to OP in a fixed ratio: determine the locus of Q.*

We shall shew that the locus is a circle.



For let C be the centre of the fixed circle; in OC take a point D such that OD is to OC in the fixed ratio, and draw any radius CP of the fixed circle; draw DQ parallel to CP meeting OP, produced if necessary, at Q. Then the  $\triangle$  OCP and ODQ are similar; [VI. 4.

$\therefore OQ : OP :: OD : OC$ , that is, in the fixed ratio.

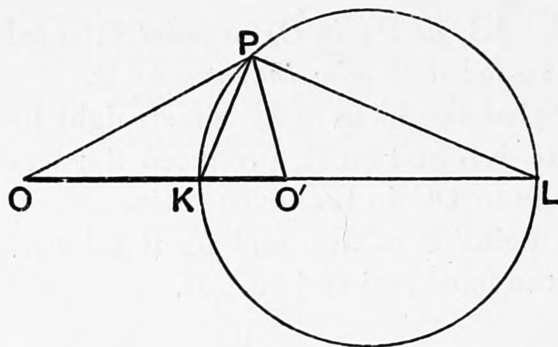
$\therefore$  Q is a point on the locus.

And DQ is to CP in the fixed ratio, so that DQ is of constant length. Hence the locus is a circle of which D is the centre.



✓ **82.** Find the locus of a point which moves so that its distances from two fixed points may be in a given ratio.

Let  $O$ ,  $O'$  be the two fixed points, and let  $P$  be any point satisfying



the given condition, so that  $OP : PO'$  in the given ratio. Bisect the interior angle  $OPO'$  by a straight line  $PK$  meeting  $OO'$  in  $K$ , and bisect the exterior angle at  $P$  by the straight line  $PL$  meeting  $OO'$  in  $L$ .

Then, by VI. 3 and A,

$$OK : KO' :: OP : PO',$$

$$\text{and } OL : LO' :: OP : PO'.$$

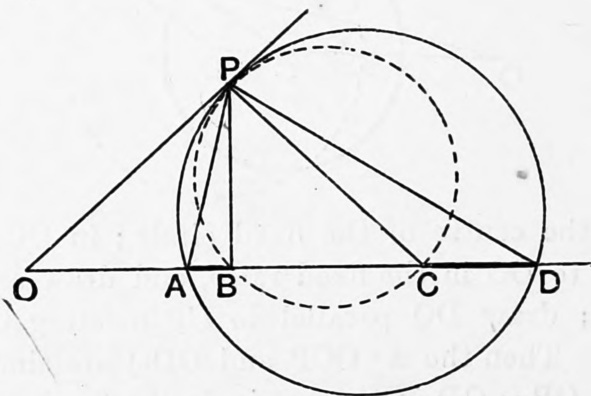
$\therefore$  the ratios  $OK : KO'$  and  $OL : LO'$  are each equal to the given ratio, and thus  $K$ ,  $L$  are both fixed.

Since  $PK$ ,  $PL$  bisect the interior and exterior angles at  $P$ ,

$\therefore KPL$  is a right angle.

$\therefore$  the locus of  $P$  is a circle described on the straight line joining the two fixed points  $K$ ,  $L$  as diameter.

✓ **83.** There are four given points  $A$ ,  $B$ ,  $C$ ,  $D$  in a straight line; required the locus of a point at which  $AB$  and  $CD$  subtend equal angles.



[We will take the case in which the points are in the following order,  $O$ ,  $A$ ,  $B$ ,  $C$ ,  $D$ .]

Let  $P$  be a point such that  $\angle APB = \angle CPD$ .

Describe a circle round the  $\triangle APD$ , and let  $PO$  be the tangent to it at  $P$  meeting the given line at  $O$ .

Then  $\angle OPA = \angle ODP$ ; [III. 32.

$$\therefore \angle OPB = \angle OPA + \angle APB = \angle ODP + \angle CPD = \angle OCP;$$

$\therefore$  by the converse of III. 32,  $OP$  is the tangent at  $P$  to the circle circumscribing the  $\triangle BPC$ , and

$$\therefore OB \cdot OC = OP^2 = OA \cdot OD.$$

It follows, as will be shewn in Art. 89, that  $O$  is a determinate point and  $OB \cdot OC$  is a known rect.;

$\therefore OP^2$  is known, and thus  $OP$  is known, and the locus of  $P$  is a circle with  $O$  as centre.

### EXERCISES.

**\*\*1.** The locus of the centroid of a triangle, whose base and area are given, is a straight line.

**\*\*2.** Given the base and vertical angle of a triangle, prove the locus of the intersection of its medians is an arc of a circle.

**\*\*3.** Prove also that its in-centre, its orthocentre, and its e-centre opposite to the given vertical angle all move on arcs of circles which pass through the extremities of the base.

**4.** If a triangle of given species (*i.e.* one whose angles are given) has one angular point fixed, and if a second moves on a given straight line, the locus of the third angular point is also a straight line.

**5.** If the second angular point move on a circle, the locus of the third is also a circle.

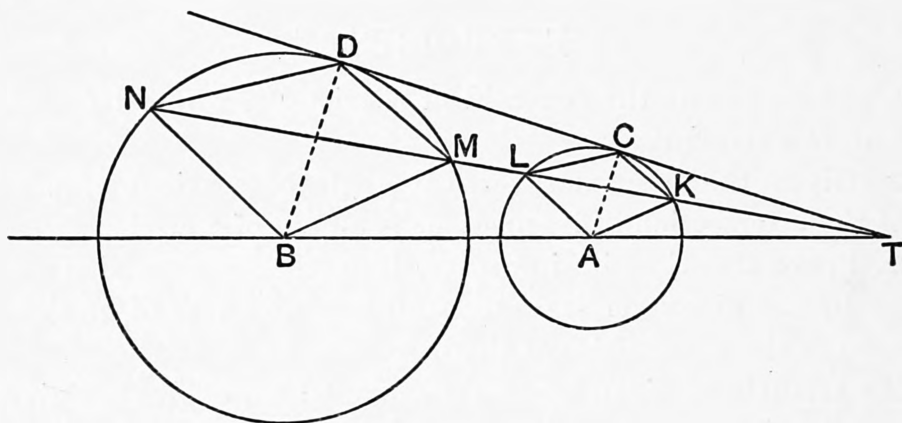
**6.** Given two points on a circle, find a third point on it whose distances from the two points are in a given ratio.

**7.** Find the locus of a point the perpendiculars from which on two given intersecting straight lines are in a given ratio.

**8.** Find a point the perpendiculars from which upon the sides of a triangle are in a given ratio

# CENTRES OF SIMILITUDE.

**84.** Let  $A$  be the centre of a circle, and  $B$  the centre of a larger circle, let a straight line be drawn touching the former circle at  $C$  and the latter circle at  $D$ , and meeting  $BA$  produced through  $A$  at  $T$ . From  $T$  draw any straight line meeting the smaller circle at  $K$  and  $L$ , and the larger circle at  $M$  and  $N$ , so that the five letters  $T, K, L, M, N$  are in this order. Then the straight lines  $AK, KC, CL, LA$  shall be respectively parallel to the straight lines  $BM, MD, DN, NB$ ; and the rectangle  $TK, TN$  shall be equal to the rectangle  $TL, TM$ , and equal to the rectangle  $TC, TD$ .



Join  $AC, BD$ . Then the  $\triangle^s TAC$  and  $TBD$  are equiangular;

$$\therefore TA : TB :: AC : BD ; \quad [\text{VI. 4.}]$$

$$:: AK : BM ;$$

$$\therefore \text{the } \triangle^s TAK, TMB \text{ are similar ;} \quad [\text{VI. 7.}]$$

$$\therefore \angle TAK = \angle TBM ;$$

$$\therefore AK, BM \text{ are parallel.}$$

Similarly  $AL, BN$  are parallel.

And because  $AK, BM$  are  $\parallel^l$  and  $AC, BD$  are  $\parallel^l$ ;

$$\therefore \angle CAK = \angle DBM ;$$

$$\therefore \angle CLK = \angle DNM ;$$

[III. 20.]

$$\therefore CL \text{ is parallel to } DN.$$

Similarly  $CK$  is parallel to  $DM$ .

$$\text{Now } TD : TN :: TM : TD ;$$

[III. 37, VI. 16.]

$$:: TK : TC ;$$

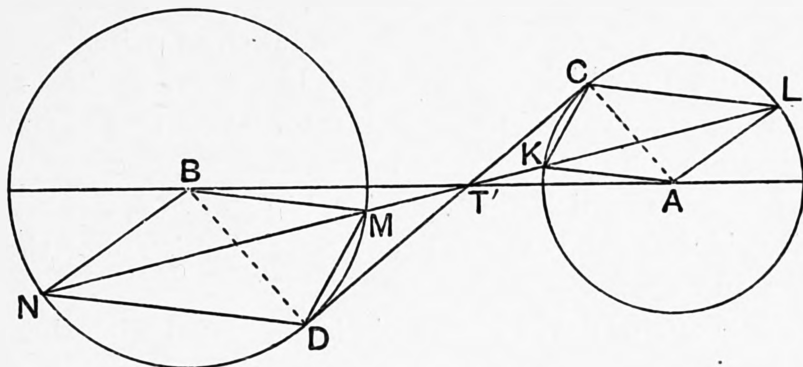
[VI. 4.]

$$\therefore \text{rect. } TC, TD = \text{rect. } TK, TN.$$

$$\text{Similarly rect. } TL, TM = \text{rect. } TC, TD.$$

If each of the given circles is without the other we may suppose the straight line which touches both circles to meet  $AB$  at  $T'$  between  $A$

and B, and the above results will all hold, provided we interchange the letters K and L; so that the five letters are now to be in the following order, L, K, T', M, N.



Since AK, BM are parallel,  $\therefore$  TK is to TM as AK to BM;  
and similarly TL is to TN as AK to BN,

*i.e.* in the ratio of the radii;

$\therefore$  all straight lines drawn through T, or T', are cut by the circles in the same constant ratio.

Hence either of the points, T or T', is called a **Centre of Similitude**, one being the external centre and the other the internal centre.

**Corollary.** Since BDT, ACT are similar  $\Delta^s$ , we have

$$BT : AT = BD : AC = \text{the ratio of the radii},$$

and similarly  $BT' : AT' = \text{the ratio of the radii}$ .

Hence the points, T and T', in which the common tangents meet the line of centres divide the line joining the centres externally and internally in the ratio of the radii.

## EXERCISES.

**\*\*1.** Two unequal similar figures of any number of sides ABCDE, A'B'C'D'E' are placed so that the corresponding sides AB and A'B', BC and B'C', etc., are parallel; the lines AA', BB', CC', etc., joining corresponding vertices all meet in the same point.

[This point is called the centre of similitude for the two figures.]

**\*\*2.** The circle of similitude of two circles is the locus of points at which the circles subtend the same angle; prove that it is the circle whose diameter is the straight line joining the centres of similitude.

[Let PQ, PQ' be two tangents to the circles, centres A, B, such that  $\angle APQ = \angle BPQ'$ , and  $\therefore$  P a point on the required circle. Since AQP, BQ'P are similar  $\Delta^s$ ,  $\therefore AP : BP = AQ : BQ' = \text{ratio of the radii}$ ;

$\therefore$  the locus of P is a circle on TT' as diameter (Art. 82).]

**\*\*3.** The orthocentre and the centroid of a triangle are the centres of similitude of the circumcircle and the nine-point circle.

**4.** The vertices of a triangle are the external centres of similitude of the in-circle and the three escribed circles, and are the internal centres of similitude for the three escribed circles taken in pairs.

**5.** The points in which the internal bisectors of the angles of a triangle meet the opposite sides are the internal centres of similitude of the in-circle and each of the escribed circles.

**6.** The points in which the exterior bisectors of the angles meet the opposite sides are the external centres of similitude of the escribed circles taken in pairs.

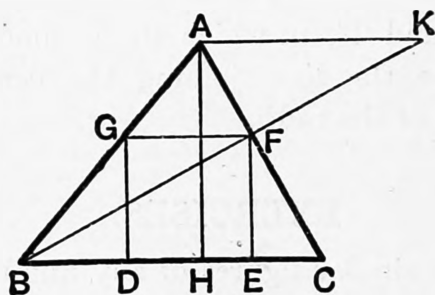
**7.** The circle of similitude of two circles is coaxal with them.

[Use Page lx, Ex. 2.]

**8.** The three circles of similitude of three circles, taken in pairs, are coaxal.

## CONSTRUCTIONS INVOLVING BOOK VI.

**85.** *To inscribe a square in a given triangle.*



Let  $ABC$  be the given  $\triangle$ , and suppose  $DEFG$  the required square. Draw  $AH$  perpendicular to  $BC$ , and  $AK$  parallel to  $BC$ ; and let  $BF$  produced meet  $AK$  at  $K$ .

Then  $BG : GF :: BA : AK$ ,

and  $BG : GD :: BA : AH$ ;

but  $GF = GD$ ;

$\therefore AH = AK$ .

[VI. 4.

[*Hypothesis.*

Hence we have the following synthetical solution. Draw  $AK$  parallel to  $BC$ , and equal to  $AH$ ; and join  $BK$ . Then  $BK$  meets  $AC$  at one of the corners of the required square, and the solution can be completed.



Suppose  $PMN$  to be the required  $\triangle$  inscribed in the given circle.

Draw  $NE$  parallel to  $AB$ ; join  $EM$ , and produce it if necessary to meet  $AB$  at  $F$ .

If the point  $F$  were known the problem might be considered solved. For  $ENM$  is a known  $\angle$ , and therefore the chord  $EM$  is known in magnitude. And then, since  $F$  is a known point, and  $EM$  is a known magnitude, the position of  $M$  becomes known. [Page 142, Ex. 12.]

We have then only to shew how  $F$  is to be determined.

The  $\angle MFA =$  the  $\angle MEN$ , [I. 29.

and  $\therefore =$  the  $\angle MPN$ ; [III. 21.

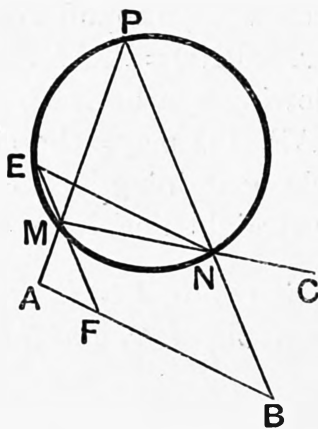
$\therefore MAF, BAP$  are similar  $\triangle^s$ ; [VI. 4.

$\therefore MA : AF :: BA : AP$ ;

$\therefore \text{rect. } MA, AP = \text{rect. } AF, BA$ .

But since  $A$  is a *given* point the rectangle  $MA, AP$  is known; and  $AB$  is known; thus  $AF$  is determined.

✓ **38.** *In a given circle it is required to inscribe a triangle so that the sides may pass through three given points,*

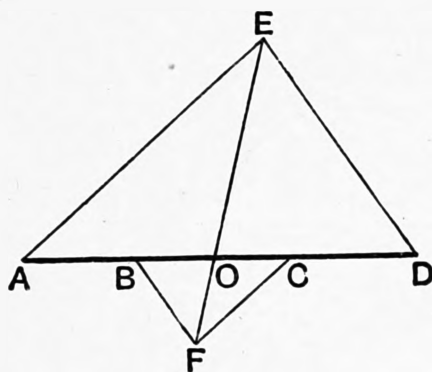


Let  $A, B, C$  be the three given points. Suppose  $PMN$  to be the required  $\triangle$  inscribed in the given circle.

Draw  $NE \parallel$  to  $AB$ , and determine the point  $F$  as in the preceding problem. We shall then have to describe in the given circle a  $\triangle EMN$  so that two of its sides may pass through given points,  $F$  and  $C$ , and the third side be  $\parallel$  to a given straight line  $AB$ . This can be done by the preceding problem.

This example and the preceding are taken from the work of Catalan already cited. The present problem is sometimes called *Castillon's* and sometimes *Cramer's*; the history of the general researches to which it has given rise will be found in a series of papers in the *Mathematician*, Vol. III., by the late T. S. Davies.

**89.** *It is required to find a point in a given straight line, such that the rectangle contained by its distances from two given points in the straight line may be equal to the rectangle contained by its distances from two other given points in the straight line.*



Let A, B, C, D be four given points in the same straight line : it is required to find a point in AD such that the rectangle contained by its distances from A and B may = the rectangle contained by its distances from C and D.

On AD describe any triangle AED ; and on CB describe a similar triangle CFB, so that CF is parallel to AE, and BF to DE ; join EF, and let it meet the given straight line at O. Then O shall be the required point.

$$\text{For, } OE : OA :: OF : OC ; \quad [\text{VI. 4.}]$$

$$\therefore OE : OF :: OA : OC. \quad [\text{V. 16.}]$$

$$\text{Similarly } OE : OF :: OD : OB ;$$

$$\therefore OA : OC :: OD : OB ; \quad [\text{V. 11.}]$$

$$\therefore \text{rect. } OA, OB = \text{rect. } OC, OD.$$

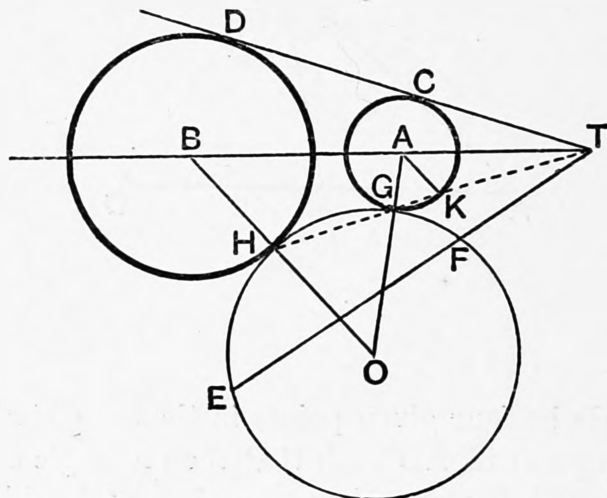
**Aliter.** Through A, B describe any circle and through C, D any other circle. Find the radical axis of these two circles. This radical axis meets the line ABCD in the required point O. For the rects. OA, OB and OC, OD are equal to the squares on the tangents from O to the circles, which are equal since O is on the radical axis.

The figure will vary slightly according to the situation of the four given points, but corresponding to an assigned situation there will be only *one* point such as is required.



**90.** *To describe a circle which shall pass through a given point and touch two given circles.*

Let A be the centre of the smaller circle and B the centre of the larger circle; and let E be the given point. Draw a straight line touching the former circle at C and the latter at D, and meeting BA, produced at T. Join TE and divide it at F so that the rect. TE, TF may = the rect. TC, TD.



Describe a circle to pass through E and F and touch either of the given circles (Art. 62); this shall be the required circle.

For suppose that the circle is described so as to touch the smaller given circle; let G be the point of contact; we have then to shew that the described circle will also touch the larger given circle. Join TG, and produce it to meet the larger given circle at H.

Then the rect. TG, TH = the rect. TC, TD [Art. 84.  
that is, = the rect. TE, TF;

$\therefore$  the described circle passes through H.

Let O be its centre, so that OGA is a straight line; we have to shew that OHB is a straight line.

Let TG intersect the smaller circle again at K; then AK is parallel to BH (Art. 84);  $\therefore \angle AKT = \angle BHG$ ,

and  $\angle AKG = \angle AGK = \angle OGH = \angle OHG$ ;

$\therefore \angle^s BHG, OHG = \angle^s AKT, AKG = \text{two right } \angle^s$ ;

$\therefore$  OHB is a straight line.

Two solutions will be obtained, because there are two solutions of the problem in Art. 62. Also, if each of the given circles is without the other, two other solutions can be obtained by taking for T the point between A and B where a straight line touching the two given circles meets AB. The various solutions correspond to the cases when the contact of circles is external or internal.

**91.** *To describe a circle which shall touch three given circles.*

Let A be the centre of that circle which is not greater than either of the other circles; let B and C be the centres of the other circles. With centre B, and radius equal to the excess of the radius of the circle B over the radius of the circle A, describe a circle. Also, with centre C, and radius equal to the excess of the radius of the circle C over the radius of the circle A, describe a circle. Describe a circle to touch externally these two described circles and to pass through A (Art. 90). Then a circle having the same centre as the last described circle, and having a radius equal to the excess of its radius over the radius of the circle A, will touch externally the three given circles.

In a similar way we may describe a circle touching internally the three given circles, or touching one of them externally and the two others internally, or touching one of them internally and the two others externally.

**EXERCISES.**

**\*\*1.** Construct a triangle given the base, the vertical angle, and the ratio of the sides.

[By Page 145 its vertex lies on a known circle, and by Art. 82 its vertex also lies on a second known circle.]

**\*\*2.** Given the base and the ratio of the sides of a triangle, construct it when

- (1) its area is given.
- (2) the sum of the squares on its sides is given.
- (3) the difference of the squares on its sides is given.

[Use Art. 82, I. 40, and Art. 32.]

Describe a triangle, having given

- 3.** its vertical  $\angle$ , its circum-radius, and the ratio of its sides.
- 4.** its vertical angle, its base, and the rectangle contained by its sides. [Use VI. C.]
- 5.** the lengths of the perpendiculars from its vertices upon the opposite sides.
- 6.** Inscribe a square in a given segment of a circle.
- 7.** Describe an equilateral triangle to be equal in area to a given isosceles triangle.
- 8.** Inscribe a regular octagon in a given square.

## ARITHMETIC, GEOMETRIC, AND HARMONIC MEANS.

**92.** The Arithmetic mean between two straight lines is a straight line which is equal to half their sum.

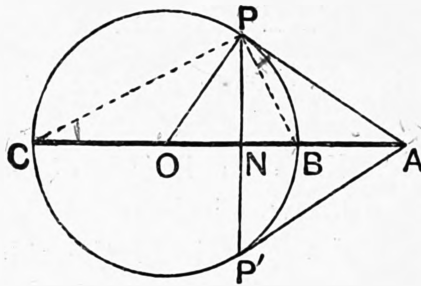
The Geometric mean is a straight line the square on which is equal to the rectangle contained by them.

The Harmonic mean is such that the first of the given straight lines is to the other as the excess of the first over the harmonic mean is to the excess of the harmonic mean over the second.

[These definitions are similar to those given in Algebra.]

**93.** *To construct these means.*

Let the two straight lines between which the means are to be found



be placed so as to have a common end, and be in the same direction as AB and AC.

Bisect BC in O. With centre O and radius OB, or OC, describe a circle, and draw tangents AP, AP' to this circle. Let PP' meet ABC in N.

Then AO is the Arithmetic, AP the Geometric, and AN the Harmonic mean between AB and AC.

For  $AC - AO = OC = BO = AO - AB$ ;

$$\therefore 2AO = AB + AC, \text{ i.e. } AO = \frac{AB + AC}{2},$$

i.e. AO is the Arithmetic mean.

Also, by III. 36,  $AP^2 = AB \cdot AC$ ,

i.e. AP is the Geometric mean.

Again, by III. 32,  $\angle APB = \angle PCB$ ;

$\therefore$  the  $\triangle^s$  APB, ACP are similar;

$$\therefore AB : BP :: AP : PC;$$

$$\therefore AB^2 : AP^2 :: BP^2 : PC^2;$$

i.e.  $AB^2 : AB \cdot AC :: BN \cdot BC : CN \cdot CB$ ; [III. 36, and VI. 8.

$$\therefore AB : AC :: BN : NC;$$

$$\therefore AC : AB :: NC : BN$$

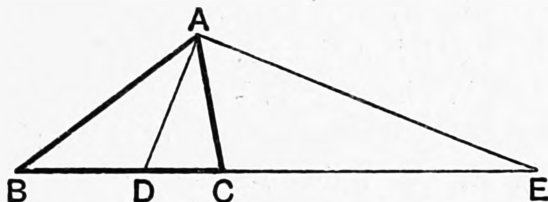
$$:: AC - AN : AN - AB;$$

$\therefore AN$  is the harmonic mean between  $AB$  and  $AC$ .

**Corollary.**—Since  $AP^2 = AN \cdot AO$ ,  $AP$  is the geometric mean between  $AN$  and  $AO$ , that is,

*The geometric mean between two given straight lines  $AB$ ,  $AC$  is also the geometric mean between the arithmetic and harmonic means between the same two straight lines.*

94. Let  $ABC$  be a triangle, and let the interior and exterior angles at  $A$  be bisected by straight lines  $AD$ ,  $AE$  which meet  $BC$  and  $BC$  produced, in  $D$  and  $E$ : then  $BD$ ,  $BC$ ,  $BE$  shall be in harmonical progression.



$$\text{For } BD : DC :: BA : AC, \quad [\text{VI. 3.}]$$

$$\text{and } BE : EC :: BA : AC; \quad [\text{VI. 4.}]$$

$$\therefore BD : DC :: BE : EC.$$

$$\therefore BE : BD :: EC : DC; \quad [\text{V. 16.}]$$

that is, as  $BE - BC$  to  $BC - BD$ .

$\therefore BD$ ,  $BC$ ,  $BE$  are in harmonical progression.

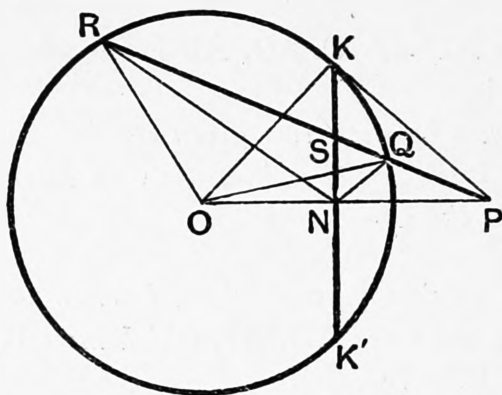
95. **Def.** When a straight line  $BC$  is divided internally and externally in the same ratio at  $D$  and  $E$ , the four points  $B$ ,  $D$ ,  $C$ ,  $E$  are said to form a **Harmonic Range**. This result is sometimes expressed by saying that  $BC$  is divided harmonically at  $D$  and  $E$ .

The two points  $B$  and  $C$  are called **Harmonic Conjugates**, and so also the two points  $D$  and  $E$ .

If  $BDCE$  is a harmonic range and the points  $B$ ,  $D$ ,  $C$ ,  $E$  be joined to a point  $A$  outside the range, the four straight lines  $AB$ ,  $AD$ ,  $AC$ ,  $AE$  are called a **Harmonic Pencil**, and either of these four lines is called a **Ray** of the Pencil; also, the point  $A$  is called the **Vertex** of the pencil.

**96.** Any straight line which passes through a fixed point is cut harmonically by the point, any circle, and the polar of the point.

Let  $P$  be the fixed point and  $PQSR$  any straight line drawn through  $P$  to cut the circle in  $Q$  and  $R$  and the polar  $KNK'$  of  $P$  in  $S$ . Then  $PQ$ ,  $PS$ ,  $PR$  shall be in harmonical progression.



Let  $KK'$  meet  $OP$  in  $N$ . Join  $OK$ ,  $OQ$ , and  $PK$ ; then  $PK$  is the tangent at  $K$  and  $\angle PKO$  is a right  $\angle$ . [Art. 42, Cor. 2.]

We have  $PN \cdot PO = PK^2$  [VI. 8.]

$= PQ \cdot PR$ ; [III. 36.]

$\therefore Q, R, O, N$  lie on a circle;

$\therefore \angle ORQ = \text{supplement of } \angle ONQ = \angle PNQ$ .

But  $\angle ORQ = \angle OQR$ , since  $OQ = OR$ .

Also  $\angle OQR = \angle ONR$  in the same segment,

since  $O, N, Q, R$  lie on a circle;

$\therefore \angle ONR = \angle PNQ$ .

$\therefore \angle RNS = \angle QNS$ , since  $ONK, PNK$  are both right angles.

$\therefore NS$  is the bisector of the  $\angle RNQ$ , and since  $KNP$  is a right  $\angle$ ,  $NP$  is the exterior bisector of the same  $\angle$ .  $\therefore$  (by Art. 94)  $PQ, PS$ , and  $PR$  are in harmonical progression.

## EXERCISES.

**1.** If  $ACBD$  be a harmonic range and  $E$  be the middle point of  $CD$ , prove that  $EA \cdot EB = EC^2$ , and conversely.

**2.** Prove also that  $AC \cdot BD = AD \cdot BC$ .

**3.** If in a harmonic pencil one ray bisect the angle between a pair of rays the fourth ray is at right angles to the first, and conversely.

[Let  $ACBD$  be a harmonic range, and  $O$  the vertex of the corresponding pencil, so that  $OC$  bisects the  $\angle AOB$  and

$$AC : CB :: AO : OB.$$

But, by the previous exercise,

$$AC : CB :: AD : DB \therefore AD : DB :: AO : OB.$$

$\therefore$  OD bisects the exterior  $\angle$  of AOB, etc.]

**4.** Two circles cut at right angles, prove that any diameter of the one is divided harmonically by the other. [Use Ex. 3, Page xxxiii.]

**5.** The circum-centre, the centroid, the nine-point centre, and the orthocentre of a triangle form a harmonic range.

**6.** Any straight line parallel to one of the rays of a harmonic pencil is divided into equal parts by the other rays, and conversely.

[Let ACBD be the harmonic range, and OA, OC, OB, OD the rays of the pencil. Through C draw NCM  $\parallel$  to OD to meet OA, OB in N, M.

Then we are given that  $AC : CB :: AD : DB$ ,

$$\text{i.e. } AC : AD :: CB : DB,$$

$$\text{i.e. } CN : DO :: CM : DO ;$$

$$\therefore CN = CM.$$

Also, by Art. 3, the median OC of the  $\triangle$  NCM bisects all lines parallel to the base NM ;  $\therefore$  etc.]

**7.** If the rays of any harmonic pencil be cut by *any* transversal, the points of intersection form a harmonic range. [Proceed as in Exercise 6.]

**8.** If four points form a harmonic range, their polars with respect to any circle give a harmonic pencil.

[For the straight lines joining the four points to the centre are, by Art. 43, inclined at the same angles as the polars of the points, etc.]

**9.** The polar circle of a triangle divides the sides harmonically.

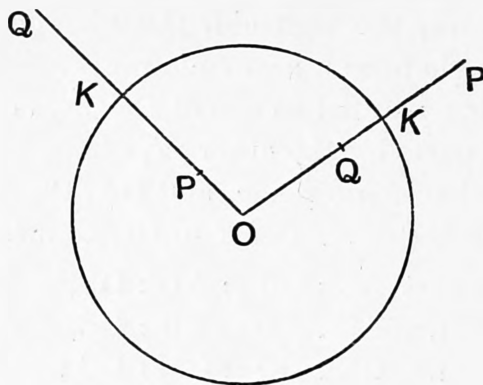
**10.** If two harmonic ranges are such that the straight lines joining three pairs of corresponding points meet in a point, the straight line joining the fourth pair passes through the same point.

[Let the two ranges be ACBD, A'C'B'D' and let AA', CC', BB' meet in O, and let OD meet A'C'B' in D<sub>1</sub>. Then, by Ex. 7, A'C'B'D<sub>1</sub> is a harmonic range ; but, by hypothesis, so also is A'C'B'D'.  $\therefore$  D' and D<sub>1</sub> coincide, etc.]

**11.** If two harmonic pencils are such that the intersections of three pairs of corresponding rays are on a straight line, the intersection of the fourth pair is on the same straight line.

## INVERSION.

**97.** Let  $O$  be the centre of a fixed circle, and let  $P$  be any point in its plane, within or without the circle. Join  $OP$ ; and let  $OP$ , pro-



duced if necessary, meet the circle in  $K$ . On  $OP$  take a point  $Q$ , such that

$$OP \cdot OQ = OK^2 = (\text{radius})^2.$$

Then  $Q$  is called the inverse of the point  $P$  with respect to the circle.

[By Art. 42 it follows that  $Q$  is the point where  $OP$  meets the polar of  $P$ .]

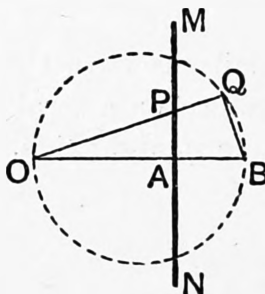
There is no real necessity for drawing the circle. If we put  $OK = k$ , then  $OQ \cdot OP = k^2$ , i.e.  $OQ = \frac{k^2}{OP}$ , and  $k^2$  may be called the constant of inversion.

The inverse of a point  $P$  with respect to a fixed point  $O$  is thus a point  $Q$  lying on  $OP$ , or  $OP$  produced, such that  $OQ = \frac{k^2}{OP}$ .

If the point  $P$  move on a certain curve, the corresponding point  $Q$  will move on another curve, and the locus of  $Q$  is called the inverse of the locus of  $P$ .

We shall find the inverse of a straight line and also that of a circle.

**98.** *To find the inverse of a straight line.*



Let  $MN$  be the straight line whose inverse with respect to  $O$  is required.

Take any point P on MN ; join OP and on it, produced if necessary, take a point Q such that  $OP \cdot OQ = k^2$ .

Draw OA perpendicular to MN and on it take a point B, such that  $OA \cdot OB = k^2$ . Join BQ.

Since  $OA \cdot OB = OP \cdot OQ$ ,

$\therefore$  A, P, Q, B lie on a circle ; [III. 36.

$\therefore \angle PAB + \angle PQB = \text{two rt. } \angle^s$ . [III. 22.

But  $\angle PAB = \text{a rt. } \angle$  ; [Construction.

$\therefore \angle PQB$ , that is  $\angle OQB$ , is a right angle.

$\therefore$  the locus of Q is a circle described on OB as diameter.

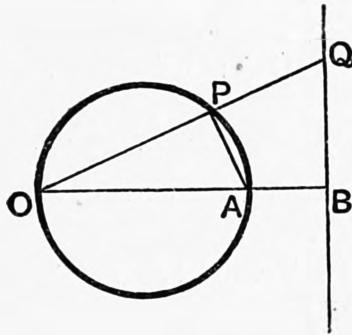
*The inverse of a straight line with respect to a fixed point O is therefore a circle, which passes through O, and whose diameter lies along the perpendicular drawn from O to the given straight line.*

Conversely, since if  $OP = \frac{k^2}{OQ}$ , then  $OQ = \frac{k^2}{OP}$ , the locus of P is obtained from that of Q by the same process by which the locus of Q is derived from that of P.

It follows that the inverse of the circle OQB with respect to O is the straight line MN.

This is more directly proved in the following theorem.

**99.** *To find the inverse of a circle with respect to a point O, which is (1) on the circumference, (2) not on the circumference.*



(1) Draw the diameter OA and on it take a point B such that  $OA \cdot OB = k^2$ , the constant of inversion. B is then a fixed point.

Join O to any point P on the curve, and on it take Q such that  $OP \cdot OQ = k^2$ . Then Q is the inverse of P.

Since  $OP \cdot OQ = OA \cdot OB$ ,

$\therefore$  P, Q, B, A lie on a circle ; [III. 36.

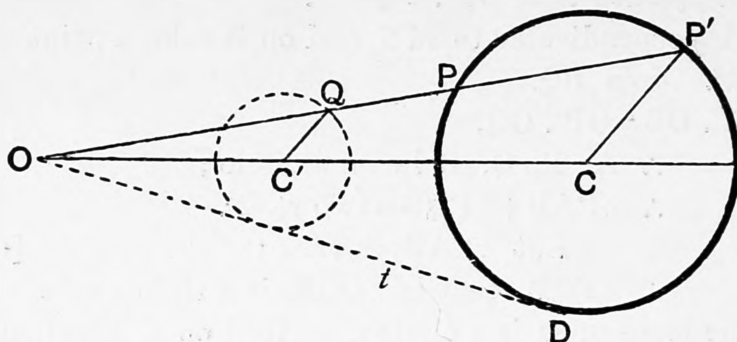
$\therefore \angle ABQ + \angle APQ = \text{two right } \angle^s$ . [III. 22.

But  $\angle APQ$  is a right  $\angle$ , [III. 31.

$\therefore \angle OBQ$  is a right angle, and thus Q lies on a straight line which is at right angles to OA and passes through the fixed point B.



(2) Let  $O$  not lie on the given circle.



Let  $C$  be the centre of the circle. Join  $O$  to any point  $P$  on the circle, and on it take  $Q$  such that  $OP \cdot OQ = k^2$ , the constant of inversion.

Let  $OP$  produced meet the circle in  $P'$ . Draw  $OD$  the tangent from  $O$  to the circle, and for simplicity put  $OD = t$ .

Then  $OP \cdot OP' = OD^2 = t^2$ ; [III. 36.]

$$\therefore OP \cdot OQ : OP \cdot OP' :: k^2 : t^2,$$

that is,  $OQ : OP' :: k^2 : t^2$ .

Draw  $QC'$  parallel to  $P'C$  to meet  $OC$  in  $C'$ .

$$\begin{aligned} \text{Then } OC' : OC &:: OQ : OP' & [\text{VI. 2.}] \\ &:: k^2 : t^2; \end{aligned}$$

$\therefore OC'$  is in a constant ratio to  $OC$ , and

$\therefore C'$  is a fixed point.

$$\begin{aligned} \text{Also } C'Q : CP' &:: OQ : OP' \\ &:: k^2 : t^2; \end{aligned}$$

$\therefore C'Q$  is constant, since  $CP'$  is constant;

$\therefore$  since  $Q$  is at a constant distance from a fixed point  $C'$  its locus is a circle whose centre is  $C'$ .

*The inverse of a circle with respect to a point outside it in its own plane is therefore a circle.*

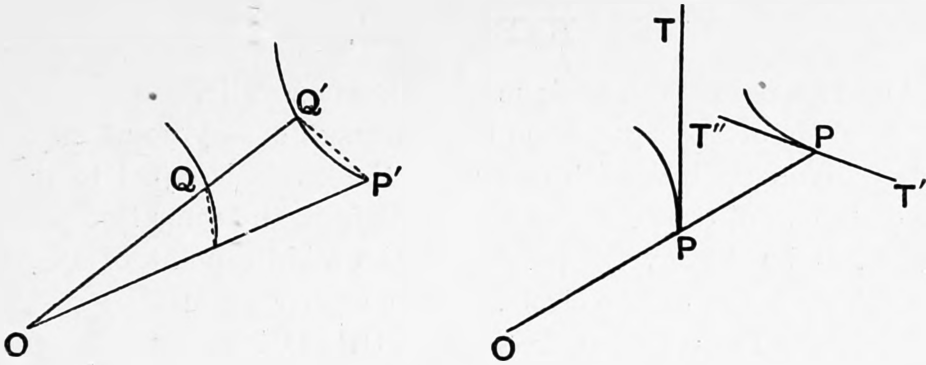
**100.** *If  $P$  be any point on a circle which is inverted with regard to a point  $O$ , and if  $P'$  be the point corresponding to  $P$ , the tangent to the first circle at  $P$  makes the same angle with  $OPP'$  that the tangent to the second circle at  $P'$  makes with  $OPP'$ .*

Let  $Q$  be a point on the first circle very close to  $P$ . Join  $OP$ ,  $OQ$ , and on them, produced if necessary, take points  $P'$ ,  $Q'$  such that

$$OP \cdot OP' = OQ \cdot OQ' = \text{constant of inversion.}$$

Then  $P'$ ,  $Q'$  are points, very close together, on the inverse circle.

Also, since  $OP \cdot OP' = OQ \cdot OQ'$ , the four points  $P$ ,  $Q$ ,  $Q'$ ,  $P'$  lie on a circle, and  $\therefore \angle OPQ = \text{supplement of } \angle QPP' = \angle OQ'P'$ . [III. 22.]



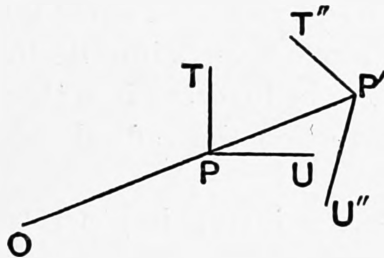
Now let  $Q$  move up indefinitely close to  $P$  and then  $PQ$  becomes the tangent at  $P$ , and the angle  $OPQ$  becomes the  $\angle$  between  $OP$  and the tangent at  $P$ , viz. the  $\angle OPT$ . Similarly  $P'Q'$  becomes the tangent at  $P'$ , and  $OQ'P'$  the angle between  $OP'$  and the tangent at  $P'$ , viz. the  $\angle OP'T'$ .

These two tangents are thus equally inclined to  $OP$ , but they are not parallel.

**Corollary 1.** The same is true if the circle  $PQ$  be replaced by a straight line or if the inverse curve  $P'Q'$  be a straight line.

**Corollary 2.** The angles which the two tangents  $PT$ ,  $P'T''$  make on the same side of  $OPP'$  [viz.  $\angle OPT$  and  $T''P'P$ ] are supplementary.

**101.** *A circle and a straight line, or two circles, intersect at the same angle as their inverses.*



Let  $P$  be a point of intersection of the first two circles, and  $PT$ ,  $PU$  their tangents at  $P$ .

Let  $P'$  be the inverse point of  $P$  and  $P'T''$ ,  $P'U''$  the tangents there.

Then  $\angle TPP' = \angle T''P'P$ ,  
and  $\angle UPP' = \angle U''P'P$ ;

$\therefore$  by addition,  $\angle TPU = \angle T''P'U''$ ;

that is, the tangents at  $P$  intersect at the same angle as the tangents at  $P'$ .

**Corollary.** If the two given circles touch at  $P$ , the angle between their tangents at  $P$  vanishes;  $\therefore$  the angle between the tangents to their inverses at  $P'$  vanishes, i.e. their inverses touch.

## EXERCISES.

1. Any two points and their inverses are concyclic. .

2. The ratio of the squares on the distances of any point on a circle from two inverse points with respect to the circle is equal to the ratio of the distances of the centre of the circle from the same two points.

[Let P, Q be the inverse points, U the middle point of PQ, O the centre, T the point on the circle, TN the perp<sup>r</sup> on OP.

$$\begin{aligned}\text{Then } TP^2 &= OT^2 + OP^2 - 2OP \cdot ON \\ &= OP \cdot OQ + OP^2 - 2OP \cdot ON \\ &= 2OU \cdot OP - 2OP \cdot ON \\ &= 2NU \cdot OP.\end{aligned}$$

$$\text{So } TQ^2 = 2NU \cdot OQ;$$

$$\therefore TP^2 : TQ^2 :: OP : OQ.]$$

3. The centre of inversion O is the centre of similitude of the original circle and its inverse.

[For CP', C'Q are parallel radii. Art. 84.]

4. Any circle can be inverted into itself.

[If the constant of inversion be taken equal to the square on the tangent OD that can be drawn from O to the circle, every such point as P inverts into P'.]

5. Any two circles can be inverted into themselves.

[Take as centre of inversion any point O on their radical axis and as constant of inversion the square on the equal tangents from O to either circle; by Art. 99 each circle inverts into itself.]

6. Any three circles can be inverted into themselves.

[Take as centre of inversion the radical centre, and proceed as in Ex. 5.]

7. A circle and a diameter invert, in general, into two circles cutting orthogonally.

[For the circle and diameter cut orthogonally and the diameter inverts into a circle; then use Art. 101.]

8. Any three circles can be inverted into three circles whose centres lie on a straight line.

[Invert with respect to any point on their orthogonal circle (Art. 79); the latter becomes a straight line which cuts the inverses of the three circles orthogonally, and the only straight line cutting three circles orthogonally passes through their centres; these latter thus lie on a straight line.]

9. Any two circles can be inverted into equal circles.

[If O be the centre of inversion and  $t, t'$  the lengths of the tangents

to the circles, of radii  $r$  and  $r'$ , then by Art. 99 (2) we must have  $t^2 : t'^2 :: r : r'$ . Hence  $O$  is a point such that the ratio of the squares of the tangents from it to the two given circles is known. Hence by Art. 79, Ex. 2,  $O$  is any point on a known circle, viz. the circle coaxal with the given ones whose centre is the external centre of similitude.]

**10.** Any three circles can be inverted into three equal circles.

**11.** When two circles intersect the circle through their common points, and the centre of inversion inverts into the radical axis of the new circles.

**12.** If  $A, B, C, D$  be four points and  $A', B', C', D'$  their inverses, then  $A'B' : AB :: k^2 : OA \cdot OB$ ,  
and  $AC \cdot BD : AB \cdot CD = A'C' \cdot B'D' : A'B' \cdot C'D'$ .

[The  $\triangle^s OB'A', OAB$  are similar ;

$$\begin{aligned} \therefore A'B' : AB &:: OB' && : OA \\ &:: OB \cdot OB' : OA \cdot OB \\ &:: k^2 && : OA \cdot OB \text{ etc.} \end{aligned}$$

**13.** Prove Euc. VI., D by inversion.

[Use Ex. 12 and the Exercise on page 91.]

**14.**  $P, Q$  are a pair of inverse points with respect to a circle  $C$ ; any circle through them cuts any diameter of the circle in a pair of inverse points.

## ON MODERN GEOMETRY.

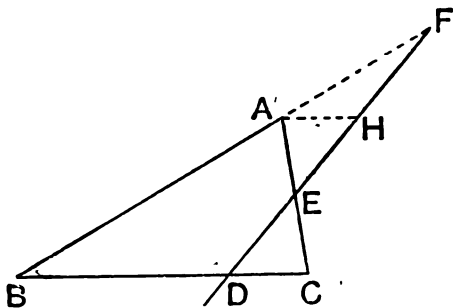
**102.** We have hitherto restricted ourselves to Euclid's Elements, and propositions which can be demonstrated by strict adherence to Euclid's methods. In modern times various other methods have been introduced, and have led to numerous and important results. These methods may be called semi-geometrical, as they are not confined within the limits of the ancient pure geometry; in fact the power of the modern methods is obtained chiefly by combining arithmetic and algebra with geometry. The student who desires to cultivate this part of mathematics may consult Townsend's *Chapters on the Modern Geometry of the Point, Line, and Circle*.

We will give as specimens some important theorems, taken from what is called the theory of transversals.

Any line, straight or curved, which cuts a system of other lines is called a **transversal**; in the examples which we shall give, the lines will be straight lines, and the system will consist of three straight lines forming a triangle.

We will give a brief enunciation of the theorem which we are about to prove, for the sake of assisting the memory in retaining the result; but the enunciation will not be fully comprehended until the demonstration is completed.

✓ **103.** *If a straight line cut the sides, or the sides produced, of a triangle, the product of three segments in order is equal to the product of the other three segments.*



Let  $ABC$  be a triangle, and let a straight line be drawn cutting the side  $BC$  at  $D$ , the side  $CA$  at  $E$ , and the side  $AB$  produced through

B at F. Then BD and DC are called *segments* of the side BC, whilst CE and EA are called *segments* of the side CA, and AF and FB are called *segments* of the side AB.

Through A draw a straight line parallel to BC, meeting DF produced at H.

Then the  $\triangle^s$  CED and EAH are equiangular ;

$$\therefore AH : CD :: AE : EC. \quad [VI. 4.]$$

$$\therefore \text{the rect. AH, EC} = \text{the rect. CD, AE.} \quad [VI. 16.]$$

Again, the  $\triangle^s$  FAH and FBD are equiangular ;

$$\therefore AH : BD :: FA : FB. \quad [VI. 4.]$$

$$\therefore \text{the rect. AH, FB} = \text{the rect. BD, FA.} \quad [VI. 16.]$$

Now suppose the straight lines represented by numbers in the manner explained in the notes to the second Book of the Elements. We have then two results which we can express arithmetically : namely,

$$\text{the product AH. EC} = \text{the product CD. AE ;}$$

$$\text{and the product AH. FB} = \text{the product BD. FA.}$$

Therefore, by the principles of Arithmetic, the product

$$\text{AH. EC. BD. FA} = \text{the product AH. FB. CD. AE,}$$

and therefore the product BD. CE. AF = the product DC. EA. FB.

This is the result intended by the enunciation given above. Each product is made by three segments, one from every side of the triangle : and the two segments which terminated at any angular point of the triangle are never in the same product.

Thus if we begin one product with the segment BD, the other segment of the side BC, namely, DC, occurs in the other product ; then the segment CE occurs in the first product, so that the two segments CD and CE, which terminate at C, do not occur in the same product ; and so on.

The student should for exercise draw another figure for the case in which the transversal meets *all* the sides *produced*, and obtain the same result.

The above theorem is generally known as **Menelaus' Theorem**, after Menelaus, an Alexandrian mathematician who lived in the 1st century after Christ.

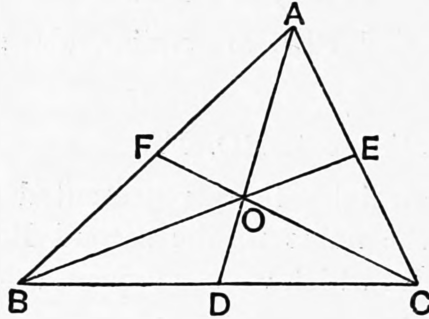
**104.** Conversely, it may be shewn by an indirect proof that if the product BD. CE. AF be equal to the product DC. EA. FB, the three points D, E, F lie in the same straight line.

✓ **105.** *If three straight lines be drawn through the angular points of a triangle to the opposite sides, and meet at the same point, the product of three segments in order is equal to the product of the other three segments.*

Let  $ABC$  be a triangle. From the angular points to the opposite sides let the straight lines  $AOD$ ,  $BOE$ ,  $COF$  be drawn, which meet at the point  $O$ : the product  $AF \cdot BD \cdot CE =$  the product  $FB \cdot DC \cdot EA$ .

For the  $\triangle ABD$  is cut by the transversal  $FOC$ ;

$$\therefore AF \cdot BC \cdot DO = FB \cdot CD \cdot OA. \quad [\text{Art. 103.}]$$



Again the  $\triangle ACD$  is cut by the transversal  $EOB$ ;

$$\therefore AO \cdot DB \cdot CE = OD \cdot BC \cdot EA. \quad [\text{Art. 103.}]$$

$\therefore$  by the principles of arithmetic,

$$AF \cdot BC \cdot DO \cdot AO \cdot DB \cdot CE = FB \cdot CD \cdot OA \cdot OD \cdot BC \cdot EA.$$

$$\therefore AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

We have supposed the point  $O$  to be within the triangle; if  $O$  be without the triangle two of the points  $D$ ,  $E$ ,  $F$  will fall on the sides produced.

Conversely, it may be shewn by an indirect proof that if the product  $AF \cdot BD \cdot CE$  be equal to the product  $FB \cdot DC \cdot EA$ , the three straight lines  $AD$ ,  $BE$ ,  $CF$  meet at the same point.

The theorem of the above article is generally known as **Ceva's Theorem**, after Ceva, in whose work, published A.D. 1678, it is first found.

## EXERCISES.

**\*\*1.** The three medians of a triangle are concurrent.

**\*\*2.** The three internal bisectors of the angles of a triangle are concurrent.

**\*\*3.** Any two external bisectors and the third internal bisector of the  $\angle^s$  of a  $\triangle$  are concurrent.

**\*\*4.** The straight lines joining each angular point to the point of contact of the opposite side of the triangle with the in-circle are concurrent.

5. ABC is a triangle, and O any point in its plane, prove that the external bisectors of the angles BOC, COA, AOB meet BC, CA, AB respectively in points lying on a straight line.

6. The in-circle of a triangle ABC touches the sides BC, CA, AB in D, E, and F; EF, FD, DE produced meet BC, CA, AB respectively in L, M, and N; prove that L, M, N are collinear.

7. Two parallelograms ACBD, A'CB'D' have a common angle at C. Prove that DD', A'B, and AB' are concurrent.

[Let DD', A'B meet in E and A'D', BD in G.]

Then  $BD \cdot GD' : A'E = DG : D'A' \cdot EB$ , [Art. 103.]

that is,  $AC \cdot BB' : A'E = AA' : B'C \cdot EB$ ,

that is,  $CA \cdot A'E \cdot BB' = AA' \cdot EB \cdot B'C$ ,

so that B'AE is a transversal of the  $\triangle A'BC$ .]

8. The line joining the points of contact of the in-circle with two sides of a triangle meets the third side in a point which, with the ends of the third side and the point of contact with the third side, form a harmonic range. [Use Menelaus' Theorem.]

9. If the tangents to the circumcircle of the  $\triangle ABC$  at its angular points meet the opposite sides in D, E, F, prove that DEF is a straight line.

$$[BD : CD = BD^2 : BD \cdot CD = BD^2 : DA^2]$$

$$= BA^2 : AC \text{ by similar triangles ; } \therefore \text{ etc.}]$$

10. ABC is a triangle inscribed in a circle, and D, E, F are the middle points of its sides BC, CA, AB; if A', B', C' are the points of the circumcircle diametrically opposite to A, B, C, prove that A'D, B'E, C'F are concurrent. [They meet in the orthocentre.]

11. If X, Y, Z are points in the sides BC, CA, AB of a  $\triangle ABC$  such that the perpendiculars to the sides at these points are concurrent, then  $BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2$ , and conversely.

12. The perpendiculars from the e-centres upon the sides of a triangle are concurrent. [Use the previous Exercise.]

13. If three concurrent straight lines through the angular points of a triangle ABC meet the opposite sides in D, E, F, and if the circle through D, E, F meet these sides in D', E', F', prove that AD', BE', CF' are also concurrent.

14. The internal and external bisectors of the angles of a triangle are drawn; their six points of intersection with the opposite sides lie, three by three, on four straight lines.

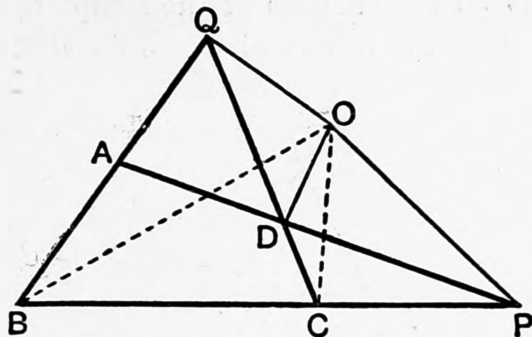
15. The six centres of similitude of three non-intersecting circles lie, three by three, on four straight lines.





**108.** *The four circles circumscribing the four triangles that can be formed by four straight lines meet in a point.*

Let the four lines be BAQ, BCP, ADP, CDQ, as in the figure.



Let the circles circumscribing the two  $\triangle^s$  QAD, DCP meet in the point O.

*Then O shall lie on the circumcircle of the  $\triangle$  BQC.*

Because D, O, Q, A lie on a circle,

$\therefore \angle DOQ = \text{the supplement of } \angle QAD \text{ [III. 22]} = \angle BAP.$

Since D, O, P, C lie on a circle,  $\therefore \angle DOC = \angle DPC$ , [III. 21.]

$\therefore \angle QOC = \angle QOD + \angle DOC = \angle BAP + \angle BPA$ ;

$\therefore \angle^s$  QOC, QBC together = the three  $\angle^s$  of the  $\triangle$  PAB = 2 rt.  $\angle^s$ ;

$\therefore$  Q, B, C, O lie on a circle, that is, O is on the circumcircle of the  $\triangle$  QBC.

Similarly it is on the circumcircle of the  $\triangle$  PAB. Hence etc.

**Corollary 1.** The feet of the perpendiculars from O on the four straight lines are collinear.

For considering the  $\triangle$  QAD, since O is on its circumcircle the feet of the perpendiculars from O on the straight lines AB, AD, CD are collinear.

So considering the  $\triangle$  QBC, the feet of the perpendiculars on the straight line AB, BC, CD are collinear.

Hence the feet of the perpendiculars on the four straight lines AB, BC, CD, DA lie on a common line.

**Corollary 2.** The orthocentres of the four are collinear.

For, since O lies on each circumcircle, the middle points of the lines joining O to the orthocentres all lie on the common pedal line. [Art. 55.]

$\therefore$  by Art. 24 the orthocentres all lie on a straight line parallel to the common pedal line.

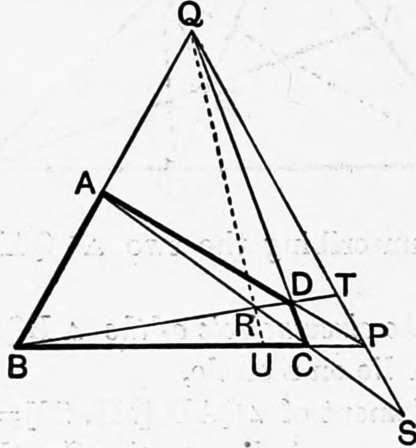
**Corollary 3.** If ABCD be a cyclic quad<sup>l</sup>, then the point O lies on the third diagonal PQ. For  $\angle POD = \angle BCQ$  and  $\angle QOD = \angle BAD$ .

$\therefore \angle POD + \angle QOD = 2 \text{ rt. } \angle^s$ , since ABCD is cyclic.

$\therefore$  POQ is a str. line.

**109.** *In a complete quadrilateral each diagonal is divided harmonically by the other two diagonals.*

Let the quadrilateral be formed by the straight lines AB, BC, CD, DA and let P, Q, R be the vertices of the complete quadrilateral; let AC, BD produced meet PQ in S and T. Join QR, and produce it to meet BP in U.



By Art. 105, since R is a point in the  $\triangle BCQ$ ,

$$\therefore BU \cdot CD \cdot QA = UC \cdot DQ \cdot AB. \dots\dots\dots(1)$$

By Art. 103, since ADP is a transversal of the same  $\triangle$ ,

$$\therefore BP \cdot CD \cdot QA = PC \cdot DQ \cdot AB. \dots\dots\dots(2)$$

From (1), (2),  $BU : BP :: UC : PC$ , i.e.  $BU : UC :: BP : PC$ .

$\therefore B, U, C, P$  form a harmonic range; [Art. 95.]

$\therefore QB, QU, QC, QP$  are a harmonic pencil; [Art. 95.]

$\therefore B, R, D, T$  is a harmonic range, [Page lxxv, Ex. 7]

and similarly  $A, R, C, S$  is a harmonic range,  
so that the diagonal BD is divided harmonically at R, T and the diagonal AC at R, S.

Also, because  $A, R, C, S$  form a harmonic range;

$\therefore BA, BR, BC, BS$  form a harmonic pencil,

and  $\therefore Q, T, P, S$  form a harmonic range, so that the diagonal QP is divided harmonically at S, T.

**Corollary.** Since  $B, R, D, T$  and  $A, R, C, S$  and also  $Q, T, P, S$  are harmonic ranges;

$\therefore$  the pencils at P, Q, R, viz. PB, PR, PA, PQ and QA, QR, QD, QP and also RQ, RD, RP, RC are harmonic pencils.

**110.** *Given three points of a harmonic range, to find the fourth.*

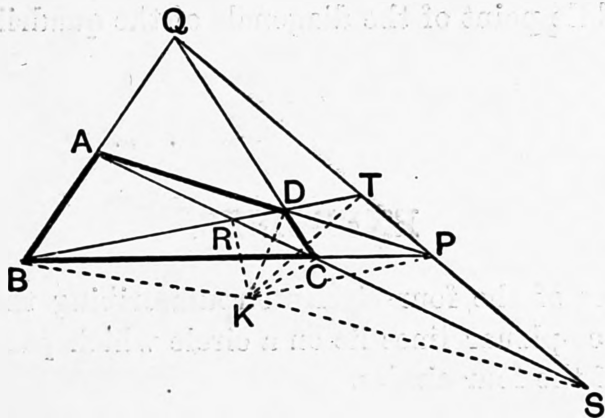
Let  $B, C, P$  be the three collinear points. Through  $P$  draw any line, and through  $B, C$  draw any two lines,  $BAQ$  and  $CDQ$ , meeting in  $Q$ , and cutting the line through  $P$  in  $A$  and  $D$ . Join  $AC$  and  $BD$  and let them meet in  $R$ . Join  $QR$  and produce it to meet  $BCP$  in  $U$ . Then by the last article  $U$  is the required fourth point of the range.

It will be noted that this construction requires the use of the ruler only.

**111.** *The circles on the three diagonals of a complete quadrilateral as diameters have a common radical axis, which is perp<sup>r</sup> to the line through the middle points of the diagonals and contains the orthocentres of the four triangles formed by taking the sides of the quadrilateral, three by three. Prove also that these three circles cut orthogonally the self-conjugate circles of these four triangles.*

Draw a complete quadrilateral as in Art. 109. Let  $K$  be one point of intersection of the circles on  $AC, BD$  as diameters.

Then  $KB, KR, KD, KT$  form a harmonic pencil and  $BKD$  is a right  $\angle$ .



$\therefore KD$  bisects  $\angle RKT$ ; [Page lxxiv, Ex. 3]

$\therefore RD : DT :: RK : KT$ .

Similarly  $KC$  bisects  $\angle RKS$ ; [Page lxxiv, Ex. 3]

$\therefore RC : CS :: RK : KS$ .

$\therefore RD \cdot KT = DT \cdot RK$  and  $RC \cdot KS = CS \cdot RK$ ;

$\therefore RD \cdot KT \cdot CS = DT \cdot RC \cdot KS$ .

But since  $CDQ$  is a transversal of  $\triangle RST$ ,

$\therefore RD \cdot TQ \cdot SC = DT \cdot QS \cdot CR$ ;

$\therefore KT : TQ :: KS : QS$ ,

i.e.  $TK : KS :: TQ : QS$ .

$\therefore KQ$  bisects  $\angle TKS$  externally.

Also  $KS, KP, KT, KQ$  is a harmonic pencil;

[Art. 109.

$\therefore PKQ$  is a right angle.

[Page lxxiv, Ex. 3]

Hence the circle on PQ as diameter goes through K, and similarly through the other point of intersection of the circles on AC, BD as diameter.

Again, the circle on PQ as diameter passes through P and the foot of the perpendicular from P on the side of BA of the  $\triangle ABP$ , i.e. it passes through two inverse points with respect to the self-conjugate circle of this  $\triangle$ . [Page xxxvi, Ex. 7]

Hence, by Art. 45, it cuts this self-conjugate circle orthogonally. Similarly for the self-conjugate circles of the other three triangles.

Similarly the circles on AC, BD cut these same four self-conjugate circles orthogonally.

Also, since the three circles on the diagonals are a coaxal system cut orthogonally by the four self-conjugate circles, the latter are a system whose centres are on the radical axis of the first three ; i.e. the four orthocentres lie on the radical axis of the circles on the diagonals as diameters, and therefore are on a straight line which is perpendicular to the line through their centres, and this is the line through the middle point of the diagonals of the quadrilateral.

## EXERCISES.

1. The centres of the four circles circumscribing the four triangles formed by four co-planar lines lie on a circle which passes through the common point of the four circles.

2. The sides of a quadrilateral inscribed in a circle are produced to meet in P and Q ; prove that (1) the bisectors of the angles at P and Q are perpendicular ; (2) the square on PQ = the sum of the squares on the tangents from P, Q to the circle ; and (3) the circle on PQ as diameter cuts the given circle orthogonally.

3. The circle described on any of the three diagonals of a complete quadrilateral as diameter cuts orthogonally the circle circumscribing the triangle formed by the three diagonals.

[Use Art. 109 and Page lxxv, Ex. 4.]

4. Draw tangents to a circle by means of a ruler only.

[Use Art. 96 and the harmonic properties of the complete quadrilateral.]

5. If a quadrilateral be inscribed in a circle and its sides be produced, and the angular points joined, so as to make a complete quadrilateral, the triangle formed by the three vertices of the latter is self-polar with respect to the circle. [Use Art. 96 and Cor. Art. 109.]

6. If ABCD (Fig. Art. 107) be a quadrilateral circumscribed about a circle, one pair of the straight lines joining the points of contact with the circle passes through the intersection of the diagonals AC, BD, and the other two pairs intersect on the third diagonal PQ.

[Let the circle touch the sides AB, BC, CD, DA in K, L, M, N. Let AC meet BD in R; let KL, MN meet in E; and KN, LM meet in F.

Then A is the pole of KN and C is the pole of LM;

$\therefore$  F is the pole of AC.

[Page xxxii, Ex. 1.]

So E is the pole of BD;

$\therefore$  R (the intersection of AC, BD) is the pole of EF.

But the intersection of KM, LN is the pole of EF;

[Ex. 5.

$\therefore$  KM, LN intersect in R.

Again, P is the pole of LN, and Q is the pole of KM;

$\therefore$  PQ is the pole of the intersection of LN and KM, *i.e.* of R;

$\therefore$  PQ and EF are each the polar of R;

$\therefore$  PQ and EF are the same str. line;  $\therefore$  E and F lie on PQ.]

7. If a quadrilateral be described about a circle, the triangle formed by the three diagonals is self-polar with respect to the circle.

[Take the figure of the previous Exercise.

Then F has been shown to be the pole of AC.

But, by Ex. 5, F is the pole of ER;

$\therefore$  AC, ER coincide, *i.e.* AC goes through E.

So BD goes through F, *i.e.* EFR is the  $\Delta$  formed by the three diagonals of ABCD, since E and F lie on PQ. [Ex. 6.

But KLMN is a quad<sup>l</sup> inscribed in the circle of which E, F, R are the vertices.

$\therefore$  EFR is a  $\Delta$  self-polar with respect to the circle.

[Ex. 5.

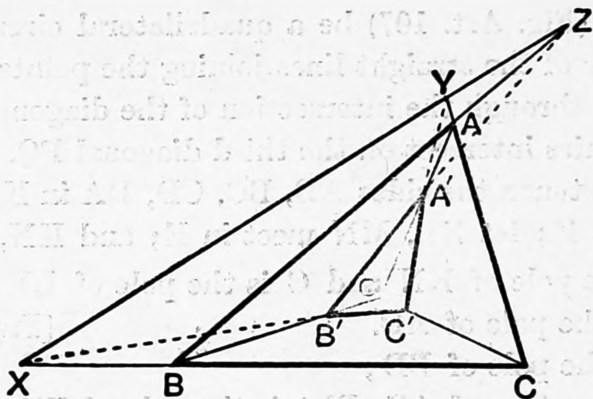
$\therefore$ , etc.]

8. If a circle be inscribed in a quadrilateral, shew that its centre is at the orthocentre of the triangle formed by the diagonals of the complete quadrilateral.

[Use the previous Exercise and Page xxxvi, Ex. 7.]

9. If a circle be inscribed in a quadrilateral its centre lies on the straight line joining the middle points of the three diagonals.

**112.** *If two triangles  $ABC$ ,  $A'B'C'$  are so placed that the straight lines  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent, then the intersections of  $BC$  and  $B'C'$ ,  $CA$  and  $C'A'$ ,  $AB$  and  $A'B'$  are collinear and conversely,*



Let  $AA'$ ,  $BB'$ ,  $CC'$  meet in  $O$ , and let  $X$ ,  $Y$ ,  $Z$  be the intersections of  $BC$  and  $B'C'$ , of  $CA$  and  $C'A'$ , and of  $AB$  and  $A'B'$ .

Since  $ZAB$  cuts the sides of the  $\triangle A'B'O$ ,

$$\therefore A'Z \cdot B'B \cdot OA = ZB' \cdot BO \cdot AA' \dots\dots\dots(1) \text{ [Art. 103.]}$$

Since  $XBC$  cuts the sides of the  $\triangle B'C'O$ ,

$$\therefore B'X \cdot C'C \cdot OB = XC' \cdot CO \cdot BB' \dots\dots\dots(2) \text{ [Art. 103.]}$$

Since  $YAC$  cuts the sides of the  $\triangle C'A'O$ ,

$$\therefore C'Y \cdot A'A \cdot OC = YA' \cdot AO \cdot CC' \dots\dots\dots(3) \text{ [Art. 103.]}$$

Multiplying together (1), (2), and (3), and cancelling like quantities on each side, we have

$$A'Z \cdot B'X \cdot C'Y = ZB' \cdot XC' \cdot YA'.$$

$\therefore X$ ,  $Y$ ,  $Z$  lie on a straight line.

[Art. 104.]

*Conversely*, if  $X$ ,  $Y$ ,  $Z$  are collinear, join  $CC'$ ,  $BB'$  and let them meet in  $O$ .

Then the triangles  $BB'Z$ ,  $CC'Y$ , have the straight lines  $BC$ ,  $B'C'$ ,  $ZY$ , joining corresponding vertices, concurrent;

$\therefore$  by the previous part of this proposition, the intersections of  $BB'$  and  $CC'$ , of  $B'Z$  and  $C'Y$ , and of  $ZB$  and  $YC$  are collinear, that is,  $O$ ,  $A'$ ,  $A$  are concurrent, that is,  $AA'$ ,  $BB'$ ,  $CC'$  meet in a point.

Two  $\triangle$ s, which are such that the lines joining corresponding vertices meet, are called *Co-polar triangles*; when their corresponding sides meet in three points which are collinear the triangles are said to be *Co-axial*. The previous theorem therefore states that *Co-polar triangles are also Co-axial and conversely*.

## EXERCISES IN GEOMETRY



## EXERCISES.

### I. 1-31.

1. In the figure of I. 2 if the diameter of the smaller circle is the radius of the larger, shew where the given point and the vertex of the constructed triangle will be situated.

2. In the figure of I. 17 shew that  $\angle ABC$  and  $\angle ACB$  are together less than two right angles, by joining  $A$  to any point in  $BC$ .

3. Two right-angled triangles have their hypotenuses equal, and a side of one equal to a side of the other: shew that they are equal in all respects.

4. If a straight line be drawn bisecting one of the angles of a triangle to meet the opposite side, the straight lines drawn from the point of section parallel to the other sides, and terminated by these sides, will be equal.

5. The side  $BC$  of a triangle  $ABC$  is produced to a point  $D$ ; the angle  $ACB$  is bisected by the straight line  $CE$  which meets  $AB$  at  $E$ . A straight line is drawn through  $E$  parallel to  $BC$ , meeting  $AC$  at  $F$ , and the straight line bisecting the exterior angle  $ACD$  at  $G$ . Shew that  $EF$  is equal to  $FG$ .

6.  $AB$  is the hypotenuse of a right-angled triangle  $ABC$ : find a point  $D$  in  $AB$  such that  $DB$  may be equal to the perpendicular from  $D$  on  $AC$ .

7.  $ABC$  is an isosceles triangle: find points  $D, E$  in the equal sides  $AB, AC$  such that  $BD, DE, EC$  may all be equal.

8. A straight line drawn at right angles to  $BC$  the base of an isosceles triangle  $ABC$  cuts the side  $AB$  at  $D$  and  $CA$  produced at  $E$ : shew that  $AED$  is an isosceles triangle.

### I. 32.

9. Shew that any angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles of the triangle taken together.

10. Construct an isosceles triangle having the vertical angle four times each of the angles at the base.

**11.** Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

**12.**  $AB$ ,  $AC$  are two straight lines given in position: it is required to find in them two points  $P$  and  $Q$ , such that,  $PQ$  being joined,  $AP$  and  $PQ$  may together be equal to a given straight line, and may contain an angle equal to a given angle.

**13.** Straight lines are drawn through the extremities of the base of an isosceles triangle, making angles with it, on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: shew that three of the triangles thus formed are isosceles.

**14.**  $AEB$ ,  $CED$  are two straight lines intersecting at  $E$ ; straight lines  $AC$ ,  $DB$  are drawn forming two triangles  $ACE$ ,  $BED$ ; the angles  $ACE$ ,  $DBE$  are bisected by the straight lines  $CF$ ,  $BF$ , meeting at  $F$ . Shew that the angle  $CFB$  is equal to half the sum of the angles  $EAC$ ,  $EDB$ .

**15.** From the angle  $A$  of a triangle  $ABC$  a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at  $D$ ; from the angle  $B$  a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at  $E$ : shew that the straight lines which join  $D$  and  $E$  to the middle point of  $AB$  are equal.

**16.** From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: shew that the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

**17.** In the figure I. 1, if  $C$  and  $F$  be the points of intersection of the circles, and  $AB$  be produced to meet one of the circles at  $K$ , shew that  $CFK$  is an equilateral triangle.

**18.** The straight lines bisecting the angles at the base of an isosceles triangle meet the sides at  $D$  and  $E$ : shew that  $DE$  is parallel to the base.

**19.**  $AB$ ,  $AC$  are two given straight lines, and  $P$  is a given point in the former: it is required to draw through  $P$  a straight line to meet  $AC$  at  $Q$ , so that the angle  $APQ$  may be three times the angle  $AQP$ .

**20.** From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

**21.** In a right-angled triangle if one of the acute angles be double the other, the hypotenuse is double the shorter side.

**22.** On the sides  $AB$ ,  $BC$  of a parallelogram  $ABCD$  equilateral triangles  $ABP$ ,  $BCQ$  are described having their vertices remote from the parallelogram. Prove that  $PQD$  is an equilateral triangle.

**23.** In the figure of I. 1, if CA and CB on being produced meet the circles again in G and H, and the circles meet again in F, then GCH is an equilateral triangle and GF and FH are in the same straight line.

**24.** In an equiangular polygon each exterior angle is one-sixth of a right angle. How many sides has it?

**25.** In an equiangular polygon each interior angle is five-thirds of a right angle. How many sides has it?

**26.** What is the least number of triangles into which a polygon of  $n$  sides can be divided?

### I. 33-34.

**27.** If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

**28.** No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

**29.** On the sides AB, BC, and CD of a parallelogram ABCD three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB, CD towards the opposite parts: shew that the distances of the vertices of the triangles on AB, CD from that on BC are respectively equal to the two diagonals of the parallelogram.

**30.** If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

**31.** Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.

**32.** ABCD is a parallelogram, and E, F, the middle points of AD and BC respectively; shew that BE and DF will trisect the diagonal AC.

**33.** Find a point D on the base BC of a triangle ABC such that, if DE, DF be drawn parallel to AC, AB respectively, AE shall be equal to AF.

**34.** The feet of the perpendiculars drawn from A upon the internal and external bisectors of the angles B, C of the triangle ABC lie upon the straight line joining the middle points of AB and AC.

**35.** ABCD is a parallelogram, X any point on BC and Y any point on AX; prove that the triangles DXY and YBC are equal.

## I. 35-45.

**36.** Construct a rhombus equal to a given parallelogram.

**37.** A straight line is drawn bisecting a parallelogram  $ABCD$  and meeting  $AD$  at  $E$  and  $BC$  at  $F$ : shew that the triangles  $EBF$  and  $CED$  are equal.

**38.** If a triangle is described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

**39.**  $D, E$  are the middle points of the sides  $AB, AC$  of a triangle, and  $CD, BE$  intersect at  $F$ : shew that the triangle  $BFC$  is equal to the quadrilateral  $ADFE$ .

**40.** The sides  $AB, AC$  of a given triangle  $ABC$  are bisected at the points  $E, F$ ; a perpendicular is drawn from  $A$  to the opposite side, meeting it at  $D$ . Shew that the angle  $FDE$  is equal to the angle  $BAC$ . Shew also that  $AFDE$  is half the triangle  $ABC$ .

**41.** Three parallelograms which are equal in all respects are placed with their equal bases in the same straight line and contiguous; the extremities of the base of the first are joined with the extremities of the side opposite to the base of the third, towards the same parts: shew that the portion of the new parallelogram cut off by the second is one half the area of any one of them.

**42.**  $ABCD$  is a parallelogram; from  $D$  draw any straight line  $DFG$  meeting  $BC$  at  $F$  and  $AB$  produced at  $G$ ; draw  $AF$  and  $CG$ : shew that the triangles  $ABF, CFG$  are equal.

**43.**  $ABC$  is a given triangle: construct a triangle of equal area, having for its base a given straight line  $AD$ , coinciding in position with  $AB$ .

**44.**  $ABC$  is a given triangle: construct a triangle of equal area, having its vertex at a given point in  $BC$  and its base in the same straight line as  $AB$ .

**45.**  $ABCD$  is a given quadrilateral: construct another quadrilateral of equal area having  $AB$  for one side, and for another a straight line drawn through a given point in  $CD$  parallel to  $AB$ .

**46.**  $ABCD$  is a given quadrilateral: construct a triangle whose base shall be in the same straight line as  $AB$ , vertex at a given point  $P$  in  $CD$ , and area equal to that of the given quadrilateral.

**47.**  $ABC$  is a given triangle: construct a triangle of equal area, having its base in the same straight line as  $AB$ , and its vertex in a given straight line.

**48.** If through the point  $O$  within a parallelogram  $ABCD$  two straight lines are drawn parallel to the sides, and the parallelograms  $OB$  and  $OD$  are equal, the point  $O$  is in the diagonal  $AC$ .

### I. 46-48.

**49.** A straight line is drawn intersecting the two sides of a right-angled triangle, and each of the acute angles is joined with the points where this straight line intersects the sides respectively opposite to them: shew that the squares on the joining straight lines are together equal to the square on the hypotenuse and the square on the straight line originally drawn.

**50.** If any point  $P$  be joined to  $A, B, C, D$ , the angular points of a rectangle, the squares on  $PA$  and  $PC$  are together equal to the squares on  $PB$  and  $PD$ .

**51.** In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn, one to bisect the opposite side, and the other perpendicular to that side, these straight lines divide the right angle into three equal parts.

**52.** On the hypotenuse  $BC$ , and the sides  $CA, AB$  of a right-angled triangle  $ABC$ , squares  $BDEC, AF$ , and  $AG$  are described: shew that the squares on  $DG$  and  $EF$  are together equal to five times the square on  $BC$ .

**53.**  $ABC$  is a triangle, right-angled at  $A$ , and the square on  $AB$  is three times that on  $AC$ ; prove that the angle  $ACB$  is twice the angle  $ABC$ .

**54.** In the figure to I. 47, prove that the triangles  $DBF, ECK, AHG, ABC$  are equal.

### I. 1-48.

**55.** From the centres  $A$  and  $B$  of two circles parallel radii  $AP, BQ$  are drawn; the straight line  $PQ$  meets the circumferences again at  $R$  and  $S$ : shew that  $AR$  is parallel to  $BS$ .

**56.** In the figure of I. 5, if the equal sides of the triangle be produced upwards through the vertex, instead of downwards through the base, a demonstration of I. 15 may be obtained without assuming any proposition beyond I. 5.

**57.** Shew that by superposition the first case of I. 26 may be immediately demonstrated, and also the second case, with the aid of I. 16.

**58.** A straight line is drawn, terminated by one of the sides of an isosceles triangle and by the other side produced, and bisected by the

base: shew that the straight lines thus intercepted between the vertex of the isosceles triangle and this straight line are together equal to the two equal sides of the triangle.

**59.** Of all parallelograms which can be formed with diagonals of given lengths the rhombus is the greatest.

**60.** If two equal straight lines intersect each other anywhere at right angles, the quadrilateral formed by joining their extremities is equal to half the square on either straight line.

**61.** Inscribe a parallelogram in a given triangle, in such a manner that its diagonals shall intersect at a given point within the triangle.

**62.** AB, AC are two given straight lines: it is required to find in AB a point P, such that if PQ be drawn perpendicular to AC, the sum of AP and AQ may be equal to a given straight line.

**63.** On a given straight line as base, construct a triangle, having given the difference of the sides and a point through which one of the sides is to pass.

**64.** Let one of the equal sides of an isosceles triangle be bisected at D, and let it also be doubled by being produced through the extremity of the base to E, then the distance of the other extremity of the base from E is double its distance from D.

**65.** Determine the locus of a point whose distance from one given point is double its distance from another given point.

**66.** A straight line AB is bisected at C, and on AC and CB as diagonals any two parallelograms ADCE and CFBG are described; let the parallelogram whose adjacent sides are CD and CF be completed, and also that whose adjacent sides are CE and CG: shew that the diagonals of these latter parallelograms are in the same straight line.

**67.** ABCD is a rectangle of which A, C are opposite angles; E is any point in BC and F is any point in CD: shew that twice the area of the triangle AEF, together with the rectangle BE, DF, is equal to the rectangle ABCD.

**68.** ABC, DBC are two triangles on the same base, and ABC has the side AB equal to the side AC; a circle passing through C and D has its centre E on CA, produced if necessary; a circle passing through B and D has its centre F on BA, produced if necessary: shew that the quadrilateral AEDF has the sum of two of its sides equal to the sum of the other two.

**69.** Two straight lines AB, AC are given in position: it is required to find in AB a point P, such that a perpendicular being drawn from it to AC, the straight line AP may exceed this perpendicular by a proposed length.

**70.** Shew that the opposite sides of any equiangular hexagon are parallel, and that any two sides which are adjacent are together equal to the two to which they are parallel.

**71.** ABC is a triangle in which C is a right angle: shew how to draw a straight line parallel to a given straight line, so as to be terminated by CA and CB, and bisected by AB.

**72.** ABC is an isosceles triangle having the angle at B four times either of the other angles; AB is produced to D so that BD is equal to twice AB, and CD is joined: shew that the triangles ACD and ABC are equiangular to one another.

**73.** Through a point K within a parallelogram ABCD straight lines are drawn parallel to the sides: shew that the difference of the parallelograms of which KA and KC are diagonals is equal to twice the triangle BKD.

**74.** Construct a right-angled triangle, having given one side and the difference between the other side and the hypotenuse.

**75.** BAC is a right-angled triangle; one straight line is drawn bisecting the right angle A, and another bisecting the base BC at right angles; these straight lines intersect at E: if D be the middle point of BC, shew that DE is equal to DA.

**76.** On AC the diagonal of a square ABCD, a rhombus AEFC is described of the same area as the square, and having its acute angle at A: if AF be joined, shew that the angle BAC is divided into three equal angles.

**77.** AB, AC are two fixed straight lines at right angles; D is any point in AB, and E is any point in AC; on DE as diagonal a half square is described with its vertex at G: shew that the locus of G is the straight line which bisects the angle BAC.

**78.** Shew that a square is greater than any other parallelogram of the same perimeter.

**79.** ABC is a triangle; AD is a third of AB, and AE is a third of AC; CD and BE intersect at F: shew that the triangle BFC is half the triangle BAC, and that the quadrilateral ADFE is equal to either of the triangles CFE or BDF.

**80.** ABC is a triangle, having the angle C a right angle; the angle A is bisected by a straight line which meets BC at D, and the angle B is bisected by a straight line which meets AC at E; AD and BE intersect at O: shew that the triangle AOB is half the quadrilateral ABDE.

**81.** Shew that a scalene triangle cannot be divided by a straight line into two parts which will coincide.



**82.** ABCD, ACED are parallelograms on equal bases BC, CE, and between the same parallels AD, BE; the straight lines BD and AE intersect at F: shew that BF is equal to twice DF.

**83.** Parallelograms AFGC, CBKH are described on AC, BC outside the triangle ABC; FG and KH meet at Z; ZC is joined, and through A and B straight lines AD and BE are drawn, both parallel to ZC, and meeting FG and KH at D and E respectively: shew that the figure ADEB is a parallelogram, and that it is equal to the sum of the parallelograms FC, CK.

**84.** If a quadrilateral have two of its sides parallel shew that the straight line drawn parallel to these sides through the intersection of the diagonals is bisected at that point.

**85.** Two triangles are on equal bases and between the same parallels: shew that the sides of the triangles intercept equal lengths of any straight line which is parallel to their bases.

**86.** In a right-angled triangle, right-angled at A, if the side AC be double of the side AB, the angle B is more than double of the angle C.

**87.** AHK is an equilateral triangle; ABCD is a rhombus, a side of which is equal to a side of the triangle, and the sides BC and CD of which pass through H and K respectively: shew that the angle A of the rhombus is ten-ninths of a right angle.

**88.** In the figure of I. 35 if two diagonals be drawn to the two parallelograms respectively, one from each extremity of the base, and the intersection of the diagonals be joined with the intersection of the sides (or sides produced) in the figure, shew that the joining straight line will bisect the base.

**89.** In the figure of I. 47 prove that AL, BK, CF meet in a point.

## II. 1-14.

**90.** A straight line is divided into two parts; shew that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

**91.** Construct a line the square on which shall be equal to the difference of two given squares.

**92.** If a straight line AB be bisected in C and produced to D so that the square on AD is three times the square on CD, and if CB be bisected in E, prove that the square on ED is three times the square on EB.



**93.** Divide a straight line into two parts so that the square on one part may equal twice the rectangle contained by the whole and the other part.

**94.** ABCDE is a straight line such that AB, BC, CD, DE are equal and O is an external point; prove that the difference of the squares on OA, OE is twice the difference of the squares on OB, OD.

**95.** In the figure of II. 11, if AB is produced to Q so that  
 $BQ = BH$ , then  $AQ^2 = 5 \cdot AH^2$ .

**96.** If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, shew that the square on the straight line so drawn will be double the square on a side of the triangle.

**97.** Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

**98.** If the angle between two adjacent sides of a parallelogram increase, while their lengths do not alter, the diagonal through their point of intersection will diminish.

**99.** Produce one side of a given triangle so that the rectangle contained by this side and the produced part may be equal to the difference of the squares on the other two sides.

**100.** Construct a rectangle equal to a given square when the sum of two adjacent sides of the rectangle is equal to a given quantity.

**101.** Construct a rectangle equal to a given square when the difference of two adjacent sides of the rectangle is equal to a given quantity.

**102.** Two rectangles have equal areas and equal perimeters: shew that they are equal in all respects.

**103.** ABCD is a rectangle; P is a point such that the sum of PA and PC is equal to the sum of PB and PD: shew that the locus of P consists of the two straight lines through the centre of the rectangle parallel to its sides.

### III. 1-15.

**104.** Two circles whose centres are A and B intersect at C; through C two chords DCE and FCG are drawn equally inclined to AB and terminated by the circles: shew that DE and FG are equal.

**105.** Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

**106.** If from any point in the diameter of a circle straight lines are drawn to the extremities of a parallel chord, the squares on these straight lines are together equal to the squares on the segments into which the diameter is divided.

**107.** A and B are two fixed points without a circle PQR; it is required to find a point P in the circumference, so that the sum of the squares described on AP and BP may be the least possible.

**108.** A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first through the point where it cuts the less circle. Shew that these two chords have the segments of the one equal to the segments of the other, each to each.

**109.** O is the centre of a circle, P is any point in its circumference, PN a perpendicular on a fixed diameter: shew that the straight line which bisects the angle OPN always passes through one or the other of two fixed points.

**110.** Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

### III. 16-19.

**111.** A circle is drawn to touch a given circle and a given straight line. Shew that the points of contact are always in the same straight line with a fixed point in the circumference of the given circle.

**112.** Draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle.

**113.** Draw a straight line cutting two given circles so that the chords intercepted within the circles shall have given lengths.

**114.** ABD, ACE are two straight lines touching a circle at B and C, and if DE be joined DE is equal to BD and CE together: shew that DE touches the circle.

**115.** Two radii of a circle at right angles to each other when produced are cut by a straight line which touches the circle: shew that the tangents drawn from the points of section are parallel to each other.

**116.** If two circles can be described so that each touches the other and three of the sides of a quadrilateral figure, then the difference

between the sums of the opposite sides is double the common tangent drawn across the quadrilateral.

**117.** AB is the diameter and C the centre of a semicircle: shew that O, the centre of any circle inscribed in the semicircle, is equidistant from C and from the tangent to the semicircle parallel to AB.

**118.** A quadrilateral is bounded by the diameter of a circle, the tangents at its extremities, and a third tangent: shew that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

**119.** If a quadrilateral, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides, shall be equal to a fourth part of the perimeter of the figure.

**120.** A series of circles touch a fixed straight line at a fixed point: shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

**121.** Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle, the sum of the two is least which make equal angles with the tangent at the point of concurrence.

**122.** C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA; join AB and produce it to meet the circle again at D, and let the tangent at D meet CB produced at E: shew that BDE is an isosceles triangle.

**123.** Let the diameter BA of a circle be produced to P, so that AP equals the radius; through A draw the tangent AED, and from P draw PEC touching the circle at C and meeting the former tangent at E; join BC and produce it to meet AED at D: then will the triangle DEC be equilateral.

**124.** If two circles touch one another externally the square on the common tangent is equal to the rectangle contained by the diameters.

**125.** ABCD is a straight line, and circles are described on AB and CD as diameters, and a common tangent to the circles is drawn meeting them in E and F. Prove that the triangles AEB and CFD are equiangular.

**126.** AB is a diameter and AP a chord of a circle; AQ is a chord bisecting the angle BAP: prove that the tangent at Q is perpendicular to AP.

**127.** The difference of the squares on the tangents drawn from any point to two concentric circles is equal to the square on the tangent drawn to the inner circle from any point on the outer circle.

## III. 20-22.

**128.** Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

**129.** Divide a circle into two parts so that the angle contained in one segment shall be equal to five times the angle contained in the other.

**130.** If the angle contained by any side of a quadrilateral and the adjacent side produced be equal to the opposite angle of the quadrilateral, shew that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

**131.** If a quadrilateral be inscribed in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

**132.** A quadrilateral can have one circle inscribed in it and another circumscribed about it; shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other.

**133.** A and B are two points on a circle centre C. An arc of a circle is drawn through A, C, and B, and a straight line APQ is drawn to cut the two circles in P and Q. Prove that PB and PQ are equal.

**134.** On the sides BC, CA, AB of a triangle any points  $\alpha$ ,  $\beta$ ,  $\gamma$  are taken. Prove that the centres of the circles circumscribing the triangles  $\beta\alpha\gamma$ ,  $\gamma\beta\alpha$ ,  $\alpha C\beta$  are the angular points of a triangle equiangular with ABC.

**135.** A and B are fixed points on a circle and N any other point on AB. Two circles are drawn, each passing through N and touching the first circle in A and B respectively. Prove that the point of intersection lies on a fixed circle.

**136.** A, B, C, D are four points on the circumference of a circle and the arcs AB, BC, CD, DA are bisected in E, F, G, H respectively; prove that EG, FH are perpendicular.

**137.** Circles are described upon the sides of any quadrilateral inscribed in a circle as diameters; prove that they intersect again in four points lying on a circle and that these four points form a quadrilateral equiangular with the given one.

**138.** Circles are described on the sides of any quadrilateral ABCD as diameters, and the four points of intersection of consecutive circles form a quadrilateral PQRS. Prove that the angles of PQRS and ABCD are equal or supplementary.

## III. 23-30.

**139.** Through a point  $C$  in the circumference of a circle two straight lines  $ACB$ ,  $DCE$  are drawn cutting the circle at  $B$  and  $E$ : shew that the straight line which bisects the angles  $ACE$ ,  $DCB$  meets the circle at a point equidistant from  $B$  and  $E$ .

**140.**  $AB$  is a diameter of a circle, and  $D$  is a given point on the circumference, such that the arc  $DB$  is less than half the arc  $DA$ : draw a chord  $DE$  on one side of  $AB$  so that the arc  $EA$  may be three times the arc  $BD$ .

**141.** From  $A$  and  $B$ , two of the angular points of a triangle  $ABC$ , straight lines are drawn so as to meet the opposite sides at  $P$  and  $Q$  in given equal angles: shew that the straight line joining  $P$  and  $Q$  will be of the same length in all triangles on the same base  $AB$ , and having vertical angles equal to  $C$ .

**142.**  $OA$ ,  $OB$ ,  $OC$  are three chords of a circle; the angle  $AOB$  is equal to the angle  $BOC$ , and  $OA$  is nearer to the centre than  $OB$ . From  $B$  a perpendicular is drawn on  $OA$ , meeting it at  $P$ , and a perpendicular on  $OC$  produced, meeting it at  $Q$ : shew that  $AP$  is equal to  $CQ$ .

**143.**  $AB$  is a given finite straight line; through  $A$  two indefinite straight lines are drawn equally inclined to  $AB$ ; any circle passing through  $A$  and  $B$  meets these straight lines at  $L$  and  $M$ . Shew that if  $AB$  be between  $AL$  and  $AM$  the sum of  $AL$  and  $AM$  is constant; if  $AB$  be not between  $AL$  and  $AM$ , the difference of  $AL$  and  $AM$  is constant.

**144.**  $AOB$  and  $COD$  are diameters of a circle at right angles to each other;  $E$  is a point in the arc  $AC$ , and  $EFG$  is a chord meeting  $COD$  at  $F$ , and drawn in such a direction that  $EF$  is equal to the radius. Shew that the arc  $BG$  is equal to three times the arc  $AE$ .

**145.** If two circles touch each other internally, any chord of the greater circle which touches the less shall be divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

**146.**  $ABCD$  is a semicircle whose diameter is  $AD$ ; the chord  $BC$  produced meets  $AD$  produced in  $E$ ; if  $CE$  is equal to the radius, prove that the arc  $AB$  is equal to three times the arc  $CD$ .

**147.** If two equal circles be drawn cutting each other in  $A$  and  $B$ , and if from  $A$  a chord is drawn cutting them in  $C$  and  $D$ , prove that the part  $CD$  between the circumferences is bisected by the circle on  $AB$  as diameter.

## III. 31.

**148.** The greatest rectangle which can be inscribed in a circle is a square.

**149.** The hypotenuse  $AB$  of a right-angled triangle  $ABC$  is bisected at  $D$ , and  $EDF$  is drawn at right angles to  $AB$ , and  $DE$  and  $DF$  are cut off each equal to  $DA$ ;  $CE$  and  $CF$  are joined: shew that the last two straight lines will bisect the angle  $C$  and its supplement respectively.

**150.** Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

**151.** Describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

**152.**  $AD$  is a diameter of a circle;  $B$  and  $C$  are points on the circumference on the same side of  $AD$ ; a perpendicular from  $D$  on  $BC$  produced through  $C$  meets it at  $E$ : shew that the square on  $AD$  is greater than the sum of the squares on  $AB$ ,  $BC$ ,  $CD$  by twice the rectangle  $BC$ ,  $CE$ .

**153.**  $AB$  is the diameter of a semicircle,  $P$  is a point on the circumference,  $PM$  is perpendicular to  $AB$ ; on  $AM$ ,  $BM$  as diameters two semicircles are described, and  $AP$ ,  $BP$  meet these latter circumferences at  $Q$ ,  $R$ : shew that  $QR$  will be a common tangent to them.

**154.**  $AB$ ,  $AC$  are two straight lines,  $B$  and  $C$  are given points in the same;  $BD$  is drawn perpendicular to  $AC$ , and  $DE$  perpendicular to  $AB$ ; in like manner  $CF$  is drawn perpendicular to  $AB$ , and  $FG$  to  $AC$ . Shew that  $EG$  is parallel to  $BC$ .

**155.** Two circles intersect at the points  $A$  and  $B$ , from which are drawn chords to a point  $C$  in one of the circumferences, and these chords, produced if necessary, cut the other circumference at  $D$  and  $E$ ; shew that the straight line  $DE$  cuts at right angles that diameter of the circle  $ABC$  which passes through  $C$ .

**156.** If squares be described on the sides and hypotenuse of a right-angled triangle, the straight line joining the intersection of the diagonals of the latter square with the right angle is perpendicular to the straight line joining the intersections of the diagonals of the two former.

**157.**  $C$  is the centre of a given circle,  $CA$  a straight line less than the radius; find the point of the circumference at which  $CA$  subtends the greatest angle.

**158.**  $AB$  is the diameter of a semicircle,  $D$  and  $E$  are any two

points in its circumference. Shew that if the chords joining A and B with D and E each way intersect at F and G, then FG produced is at right angles to AB.

**159.** Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other: shew that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

**160.** A circle is described on the shorter diagonal of a rhombus as a diameter, and cuts the sides; and the points of intersection are joined crosswise with the extremities of that diagonal: shew that the parallelogram thus formed is a rhombus with angles equal to those of the first.

**161.** Through a fixed point O a straight line is drawn, cutting a fixed circle in P and Q, and upon OP, OQ as chords are described circles touching the fixed circles in P and Q. Prove that the two circles so described will intersect on another fixed circle.

**162.** Two segments of circles are on the same base AB, and P is any point on one of them; the straight lines APD, BPC are drawn meeting the circumferences of the other segment in D and C; AC and BD meet in Q. Prove that the angle AQB is constant.

**163.** Any point P is taken on the given segment of a circle described on a line AB, and AG and BH are let fall upon BP and AP respectively; prove that GH touches a fixed circle.

**164.** A straight line AD is trisected at BC. On AB, BD as diameters circles are described; shew how to draw a straight line through A so that the parts of it intercepted by the two circles may be equal in length.

**165.** Through a fixed point A on a given circle a line is drawn cutting the circle at P, and AP is produced to Q, so that PQ is constant. Prove that the straight line through Q, perpendicular to AQ, touches a fixed circle.

**166.** AB is the diameter of a circle, and AC, AD two chords which when produced meet the tangent at B in E and F. Prove that a circle will pass through C, D, E, F.

### III. 32-34.

**167.** If two circles touch each other any straight line drawn through the point of contact will cut off similar segments.

**168.** AB is any chord, and AD is a tangent to a circle at A. DPQ is any straight line parallel to AB, meeting the circumference at P and Q. Shew that the triangle PAD is equiangular to the triangle QAB.



**169.** Two circles  $ABDH$ ,  $ABG$ , intersect each other at the points  $A$ ,  $B$ ; from  $B$  a straight line  $BD$  is drawn in the one to touch the other; and from  $A$  any chord whatever is drawn cutting the circles at  $G$  and  $H$ : shew that  $BG$  is parallel to  $DH$ .

**170.** Two circles intersect at  $A$  and  $B$ . At  $A$  the tangents  $AC$ ,  $AD$  are drawn to each circle and terminated by the circumference of the other. If  $CB$ ,  $BD$  be joined, shew that  $AB$  or  $AB$  produced, if necessary, bisects the angle  $CBD$ .

**171.**  $AB$  is any chord of a circle,  $P$  any point on the circumference of the circle;  $PM$  is a perpendicular on  $AB$  and is produced to meet the circle at  $Q$ , and  $AN$  is drawn perpendicular to the tangent at  $P$ : shew that the triangle  $NAM$  is equiangular to the triangle  $PAQ$ .

**172.** Two diameters  $AOB$ ,  $COD$  of a circle are at right angles to each other;  $P$  is a point in the circumference; the tangent at  $P$  meets  $COD$  produced at  $Q$ , and  $AP$ ,  $BP$  meet the same line at  $R$ ,  $S$  respectively: shew that  $RQ$  is equal to  $SQ$ .

**173.** From a given point  $A$  without a circle, whose centre is  $O$ , draw a straight line cutting the circle at the points  $B$  and  $C$ , so that the area  $BOC$  may be the greatest possible.

**174.** Find a point within a triangle at which the three sides subtend equal angles.

**175.**  $ABC$  is a triangle inscribed in a circle, and  $AP$ ,  $BQ$  are drawn parallel to  $BC$ ,  $CA$  respectively to meet the circle in  $P$  and  $Q$ ; prove that the chord  $PQ$  is parallel to the tangent to the circle at  $C$ .

**176.** Two circles  $PQA$ ,  $PQB$  cut at  $P$  and  $Q$ . The tangent at  $P$  to the circle  $PQB$  meets the circle  $PQA$  at  $A$ , and the tangent at  $P$  to  $PQA$  meets  $PQB$  at  $B$ . Prove that the angles  $PQA$ ,  $PQB$  are equal.

**177.**  $PMT$  is a tangent to the circle  $APC$  at the point  $P$ ;  $CNAT$  is a diameter to which  $PN$  is drawn perpendicular and  $AM$  is perpendicular to  $PT$ ; prove that  $AM$ ,  $AN$  are equal.

**178.** In a right-angled triangle, if a semicircle be described on one of the sides, the tangent at the point where it cuts the hypotenuse bisects the other side.

**179.** The diagonals of a cyclic quadrilateral  $ABCD$  intersect in  $E$ ; prove that the tangent at  $E$  to the circle about the triangle  $ABE$  is parallel to  $CD$ .

### III. 35-37.

**180.** Two circles  $ABCD$ ,  $EBCF$ , having the common tangents  $AE$  and  $DF$ , cut one another at  $B$  and  $C$ , and the chord  $BC$  is produced to cut the tangents at  $G$  and  $H$ : shew that the square on  $GH$  exceeds the square on  $AE$  or  $DF$  by the square on  $BC$ .



**181.**  $ABC$  is a right-angled triangle; from any point  $D$  in the hypotenuse  $BC$  a straight line is drawn at right angles to  $BC$ , meeting  $CA$  at  $E$  and  $BA$  produced at  $F$ : shew that the square on  $DE$  is equal to the difference of the rectangles  $BD$ ,  $DC$  and  $AE$ ,  $EC$ ; and that the square on  $DF$  is equal to the sum of the rectangles  $BD$ ,  $DC$  and  $AF$ ,  $FB$ .

**182.** It is required to find a point in the straight line which touches a circle at the end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

### III. 1-37.

**183.**  $AD$ ,  $BE$  are perpendiculars from the angles  $A$  and  $B$  of a triangle on the opposite sides;  $BF$  is perpendicular to  $ED$  or  $DE$  produced: shew that the angle  $FBD$  is equal to the angle  $EBA$ .

**184.** If  $ABC$  be a triangle, and  $BE$ ,  $CF$  the perpendiculars from the angles on the opposite sides, and  $K$  the middle point of the third side, shew that the angles  $FEK$ ,  $EFK$  are each equal to  $A$ .

**185.**  $AB$  is a diameter of a circle;  $AC$  and  $AD$  are two chords meeting the tangent at  $B$  at  $E$  and  $F$  respectively: shew that the angles  $FCE$  and  $FDE$  are equal.

**186.** Two circles cut one another at a point  $A$ : it is required to draw through  $A$  a straight line so that the extreme length of it intercepted by the two circles may be equal to that of a given straight line.

**187.** Draw from a given point in the circumference of a circle, a chord which shall be bisected by its point of intersection with a given chord of the circle.

**188.** When an equilateral polygon is described about a circle it must necessarily be equiangular if the number of sides be odd, but not otherwise.

**189.** If any number of triangles on the same base  $BC$ , and on the same side of it have their vertical angles equal, and perpendiculars, intersecting at  $D$ , be drawn from  $B$  and  $C$  on the opposite sides, find the locus of  $D$ ; and shew that all the straight lines which bisect the angle  $BDC$  pass through the same point.

**190.** Let  $O$  and  $C$  be any fixed points on the circumference of a circle, and  $OA$  any chord; then if  $AC$  be joined and produced to  $B$ , so that  $OB$  is equal to  $OA$ , the locus of  $B$  is an equal circle.

**191.** From any point  $P$  in the diagonal  $BD$  of a parallelogram  $ABCD$ , straight lines  $PE$ ,  $PF$ ,  $PG$ ,  $PH$  are drawn perpendicular to the sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ : shew that  $EF$  is parallel to  $GH$ .

**192.** Through any fixed point of a chord of a circle other chords are drawn: shew that the straight lines from the middle point of the first chord to the middle points of the others will meet them all at the same angle.

**193.**  $ABC$  is a straight line, divided at any point  $B$  into two parts;  $ADB$  and  $CDB$  are similar segments of circles, having the common chord  $BD$ ;  $CD$  and  $AD$  are produced to meet the circumferences at  $F$  and  $E$  respectively, and  $AF$ ,  $CE$ ,  $BF$ ,  $BE$  are joined: shew that  $ABF$  and  $CBE$  are isosceles triangles, equiangular to one another.

**194.**  $A$  is a given point: it is required to draw from  $A$  two straight lines which shall contain a given angle and intercept on a given straight line a part of given length.

**195.**  $A$  and  $B$  are the centres of two circles which touch internally at  $C$ , and also touch a third circle, whose centre is  $D$ , externally and internally respectively at  $E$  and  $F$ : shew that the angle  $ADB$  is double of the angle  $ECF$ .

**196.**  $C$  is the centre of a circle, and  $CP$  is the perpendicular on a chord  $APB$ : shew that the sum of  $CP$  and  $AP$  is greatest when  $CP$  is equal to  $AP$ .

**197.**  $AB$ ,  $BC$ ,  $CD$  are three adjacent sides of any polygon inscribed in a circle; the arcs  $AB$ ,  $BC$ ,  $CD$  are bisected at  $L$ ,  $M$ ,  $N$ ; and  $LM$  cuts  $BA$ ,  $BC$  respectively at  $P$  and  $Q$ : shew that  $BPQ$  is an isosceles triangle, and that the angles  $ABC$ ,  $BCD$  are together double of the angle  $LMN$ .

**198.** In the circumference of a given circle determine a point so situated that if chords be drawn to it from the extremities of a given chord of the circle their difference shall be equal to a given straight line less than the given chord.

**199.** Construct a triangle, having given the sum of the sides, the difference of the segments of the base made by the perpendicular from the vertex, and the difference of the base angles.

**200.**  $AKL$  is a fixed straight line cutting a given circle at  $K$  and  $L$ ;  $APQ$ ,  $ARS$  are two other straight lines making equal angles with  $AKL$ , and cutting the circle at  $P$ ,  $Q$  and  $R$ ,  $S$ : shew that whatever be the position of  $APQ$  and  $ARS$ , the straight line joining the middle points of  $PQ$  and  $RS$  always remains parallel to itself.

**201.** If about a quadrilateral another quadrilateral can be described such that every two of its adjacent sides are equally inclined to that

side of the former quadrilateral which meets them both, then a circle may be described about the former quadrilateral.

**202.** Two circles touch one another internally at the point  $A$ : it is required to draw from  $A$  a straight line such that the part of it between the circles may be equal to a given straight line, which is not greater than the difference between the diameters of the circles.

**203.**  $ABCD$  is a parallelogram;  $AE$  is at right angles to  $AB$ , and  $CE$  is at right angles to  $CB$ : shew that  $ED$ , if produced, will cut  $AC$  at right angles.

**204.** The two angles at the base of a triangle are bisected by two straight lines on which perpendiculars are drawn from the vertex: shew that the straight line which passes through the feet of these perpendiculars will be parallel to the base and will bisect the sides.

**205.** In a given circle inscribe a rectangle equal to a given rectilineal figure.

**206.** In an acute-angled triangle  $ABC$  perpendiculars  $AD$ ,  $BE$  are let fall on  $BC$ ,  $CA$  respectively; circles described on  $AC$ ,  $BC$  as diameters meet  $BE$ ,  $AD$  respectively at  $F$ ,  $G$  and  $H$ ,  $K$ : shew that  $F$ ,  $G$ ,  $H$ ,  $K$  lie on the circumference of a circle.

**207.** Two diameters in a circle are at right angles; from their extremities four parallel straight lines are drawn: shew that they divide the circumference into four equal parts.

**208.**  $E$  is the middle point of a semicircle arc  $AEB$ , and  $CDE$  is any chord cutting the diameter at  $D$ , and the circle at  $C$ : shew that the square on  $CE$  is twice the quadrilateral  $AEBC$ .

**209.**  $AB$  is a fixed chord of a circle,  $AC$  is a moveable chord of the same circle; a parallelogram is described of which  $AB$  and  $AC$  are adjacent sides: find the locus of the middle points of the diagonals of the parallelogram.

**210.**  $AB$  is a fixed chord of a circle,  $AC$  is a moveable chord of the same circle; a parallelogram is described of which  $AB$  and  $AC$  are adjacent sides: determine the greatest possible length of the diagonal drawn through  $A$ .

**211.** If two equal circles be placed at such a distance apart that the tangent drawn to either of them from the centre of the other is equal to a diameter, shew that they will have a common tangent equal to the radius.

**212.** Find a point in a given circle from which if two tangents be drawn to an equal circle, given in position, the chord joining the points of contact is equal to the chord of the first circle formed by joining the points of intersection of the two tangents produced; and determine the limit to the possibility of the problem.

**213.**  $AB$  is a diameter of a circle, and  $AF$  is any chord;  $C$  is any point in  $AB$ , and through  $C$  a straight line is drawn at right angles to  $AB$ , meeting  $AF$ , produced if necessary at  $G$ , and meeting the circumference at  $D$ : shew that the rectangle  $FA, AG$ , and the rectangle  $BA, AC$ , and the square on  $AD$  are all equal.

**214.**  $A, B, C$  are three given points in the circumference of a given circle: find a point  $P$  such that if  $AP, BP, CP$  meet the circumference at  $D, E, F$  respectively, the arcs  $DE, EF$  may be equal to given arcs.

**215.** Find the point in the circumference of a given circle, the sum of whose distances from two given straight lines at right angles to each other, which do not cut the circle, is the greatest or least possible.

**216.** On the sides of a triangle segments of a circle are described *internally*, each containing an angle equal to the excess of two right angles above the opposite angle of the triangle: shew that the radii of the circles are equal, that the circles all pass through one point, and that their chords of intersection are respectively perpendicular to the opposite sides of the triangle.

#### IV. 1-4.

**217.** In IV. 3 shew that the straight lines drawn through  $A$  and  $B$  to touch the circle will meet.

**218.** In IV. 4 shew that the straight lines which bisect the angles  $B$  and  $C$  will meet.

**219.** In IV. 4 shew that the straight line  $DA$  will bisect the angle at  $A$ .

**220.** Two opposite sides of a quadrilateral are together equal to the other two, and each of the angles is less than two right angles. Shew that a circle can be inscribed in the quadrilateral.

**221.** Two circles  $HPL, KPM$ , that touch each other externally, have the common tangents  $HK, LM$ ;  $HL$  and  $KM$  being joined, shew that a circle may be inscribed in the quadrilateral  $HKML$ .

**222.** Given the base, the vertical angle, and the radius of the inscribed circle of a triangle, construct it.

#### IV. 5-9.

**223.** Shew that if the straight line joining the centres of the inscribed and circumscribed circles of a triangle passes through one of its angular points, the triangle is isosceles.

**224.** The common chord of two circles is produced to any point  $P$ ;  $PA$  touches one of the circles at  $A$ ,  $PBC$  is any chord of the other.

Shew that the circle which passes through A, B, and C touches the circle to which PA is a tangent.

**225.** Describe a circle which shall pass through two given points and cut off from a given straight line a chord of given length.

**226.** Describe a circle which shall have its centre in a given straight line, and cut off from two given straight lines chords of equal given length.

**227.** Describe a circle which shall pass through two given points, so that the tangent drawn to it from another given point may be of a given length.

**228.** C is the centre of a circle; CA, CB are two radii at right angles; from B any chord BP is drawn cutting CA at N; prove that the circle described about the triangle ANP will be touched by BA.

**229.** The angle ACB of any triangle is bisected, and the base AB is bisected at right angles, by straight lines which intersect at D: shew that the angles ACB, ADB are together equal to two right angles.

**230.** ACDB is a semicircle, AB being the diameter, and the two chords AD, BC intersect at E: shew that if a circle be described round CDE it will cut the former at right angles.

**231.** A circle is described round the triangle ABC; the tangent at C meets AB produced at D; the circle whose centre is D and radius DC cuts AB at E: shew that EC bisects the angle ACB.

**232.** AB, AC are two straight lines given in position; BC is a straight line of given length; D, E are the middle points of AB, AC; DF, EF are drawn at right angles to AB, AC respectively. Shew that AF will be constant for all positions of BC.

**233.** A circle is described about an isosceles triangle ABC in which AB is equal to AC; from A a straight line is drawn meeting the base at D and the circle at E: shew that the circle which passes through B, D, and E, touches AB.

**234.** AC is a chord of a given circle; B and D are two given points in the chord, both within or both without the circle: if a circle be described to pass through B and D, and touch the given circle, shew that AB and CD subtend equal angles at the point of contact.

**235.** A and B are two points within a circle: find the point P in the circumference such that if PAH, PBK be drawn meeting the circle at H and K, the chord HK shall be the greatest possible.

**236.** The centre of a given circle is equidistant from two given straight lines: describe another circle which shall touch these two straight lines and shall cut off from the given circle a segment containing an angle equal to a given angle.

## IV. 10-16.

**237.** On a given straight line as base describe an isosceles triangle having the third angle treble of each of the angles at the base.

**238.** If  $A$  be the vertex and  $BD$  the base of the constructed triangle in IV. 10,  $D$  being one of the two points of intersection of the two circles employed in the construction, and  $E$  the other, and  $AE$  be drawn meeting  $BD$  produced at  $G$ , shew that  $GAB$  is another isosceles triangle of the same kind.

**239.** In the figure of IV. 10 if the two equal chords of the smaller circle be produced to cut the larger, and these points of section be joined, another triangle will be formed having the property required by the proposition.

**240.** In the figure of IV. 10 if  $AF$  be the diameter of the smaller circle,  $DF$  is equal to a radius of the circle which circumscribes the triangle  $BCD$ .

**241.** Shew that each of the triangles made by joining the extremities of adjoining sides of a regular pentagon is less than a third and greater than a fourth of the whole area of the pentagon.

**242.** Shew how to derive a regular hexagon from an equilateral triangle inscribed in a circle, and from the construction shew that the side of the hexagon equals the radius of the circle, and that the hexagon is double of the triangle.

**243.** In a given circle inscribe a triangle whose angles are as the numbers 2, 5, 8.

**244.** If  $ABCDEF$  is a regular hexagon, and  $AC$ ,  $BD$ ,  $CE$ ,  $DF$ ,  $EA$ ,  $FB$  be joined, another regular hexagon is formed whose area is one third of that of the former.

## IV. 1-16.

**245.** The points of contact of the inscribed circle of a triangle are joined; and from the angular points of the triangle so formed perpendiculars are drawn to the opposite sides: shew that the triangle of which the feet of these perpendiculars are the angular points has its sides parallel to the sides of the original triangle.

**246.** Construct a triangle having given an angle and the radii of the inscribed and circumscribed circles.

**247.**  $ABCDE$  is a regular pentagon; join  $AC$  and  $BD$  intersecting at  $O$ : shew that  $AO$  is equal to  $DO$ , and that the rectangle  $AC$ ,  $CO$  is equal to the square on  $BC$ .

**248.** A straight line  $PQ$  of given length moves so that its ends are always on two fixed straight lines  $CP$ ,  $CQ$ ; straight lines from  $P$  and  $Q$  at right angles to  $CP$  and  $CQ$  respectively intersect at  $R$ ; perpendiculars from  $P$  and  $Q$  on  $CQ$  and  $CP$  respectively intersect at  $S$ : shew that the loci of  $R$  and  $S$  are circles having their common centre at  $C$ .

**249.** Right-angled triangles are described on the same hypotenuse: shew that the locus of the centres of the inscribed circles is a quarter of the circumference of a circle of which the common hypotenuse is a chord.

**250.** On a given straight line  $AB$  any triangle  $ACB$  is described; the sides  $AC$ ,  $BC$  are bisected and straight lines drawn at right angles to them through the points of bisection to intersect at a point  $D$ ; find the locus of  $D$ .

**251.** Construct a triangle, having given its base, one of the angles at the base, and the distance between the centre of the inscribed circle and the centre of the circle touching the base and the sides produced.

**252.** Within a given circle inscribe three equal circles, touching one another and the given circle.

**253.** If the radius of a circle be cut as in II. 11, the square on its greater segment, together with the square on the radius is equal to the square on the side of a regular pentagon inscribed in the circle.

**254.** From the vertex of a triangle draw a straight line to the base so that the square on the straight line may be equal to the rectangle contained by the segments of the base.

**255.** The perpendiculars from the angles  $A$  and  $B$  of a triangle  $ABC$  on the opposite sides meet at  $D$ ; the circles described round  $ADC$  and  $DBC$  cut  $AB$  or  $AB$  produced at the points  $E$  and  $F$ : shew that  $AE$  is equal to  $BF$ .

**256.** Four circles are described so that each may touch internally three of the sides of a quadrilateral: shew that a circle may be described so as to pass through the centres of the four circles.

**257.** A circle is described round the triangle  $ABC$ , and from any point  $P$  of its circumference perpendiculars are drawn to  $BC$ ,  $CA$ ,  $AB$ , which meet the circle again at  $D$ ,  $E$ ,  $F$ : shew that the triangles  $ABC$  and  $DEF$  are equal in all respects, and that the straight lines  $AD$ ,  $BE$ ,  $CF$  are parallel.

**258.** With any point in the circumference of a given circle as centre, describe another circle, cutting the former at  $A$  and  $B$ ; from  $B$  draw in the described circle a chord  $BD$  equal to its radius, and join  $AD$ , cutting the given circle at  $Q$ : shew that  $QD$  is equal to the radius of the given circle.



**259.** A point is taken without a square, such that straight lines being drawn to the angular points of the square, the angle contained by the two extreme straight lines is divided into three equal parts by the other two straight lines : shew that the locus of the point is the circumference of the circle circumscribing the square.

**260.** Circles are inscribed in the two triangles formed by drawing a perpendicular from an angle of a triangle on the opposite side ; and analogous circles are described in relation to the two other like perpendiculars : shew that the sum of the diameters of the six circles together with the sum of the sides of the original triangle is equal to twice the sum of the three perpendiculars.

**261.** Three concentric circles are drawn in the same plane : draw a straight line, such that one of its segments between the inner and outer circumference may be bisected at one of the points at which the straight line meets the middle circumference.

## VI. 1-A.

**262.** Shew that one of the triangles in the figure of IV. 10 is a mean proportional between the other two.

**263.** Find a point within a triangle such that if straight lines be drawn from it to the three angular points the triangle will be divided into three triangles equal in area.

**264.** ABC is a triangle ; any straight line parallel to BC meets AB at D and AC at E ; join BE and CD meeting at F : shew that the triangle ADF is equal to the triangle AEF.

**265.** ABC is a triangle ; any straight line parallel to BC meets AB at D and AC at E ; join BE and CD meeting at F : shew that if AF be produced it will bisect BC.

**266.** AB is a diameter of a circle, CD is a chord at right angles to it, and E is any point in CD ; AE and BE are drawn and produced to cut the circle at F and G : shew that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the remaining two.

**267.** Apply VI. 3 to solve the problem of the trisection of a finite straight line.

**268.** Three points D, E, F in the sides of a triangle ABC being joined form a second triangle, such that any two sides make equal angles with the side of the former at which they meet : shew that AD, BE, CF are at right angles to BC, CA, AB respectively.



**269.** D, E, and F are the middle points of the sides BC, CA, AB of a triangle ABC. Through A a straight line is drawn cutting DF, DE in M and N respectively. Prove that BM and CN are parallel.

**270.** AB is a straight line given in magnitude and direction and D is any point in it. EF is another given straight line unlimited in length. Find a point P on it such that DP bisects the angle APB.

## VI. 4-6.

**271.** If two triangles be on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.

**272.** A tangent to a circle at the point A intersects two parallel tangents at B, C, the points of contact of which with the circle are D, E respectively; and BE, CD intersect at F: shew that AF is parallel to the tangents BD, CE.

**273.** P and Q are fixed points; AB and CD are fixed parallel straight lines; any straight line is drawn from P to meet AB at M, and a straight line is drawn from Q parallel to PM meeting CD at N: shew that the ratio of PM to QN is constant, and thence shew that the straight line through M and N passes through a fixed point.

**274.** A and B are two points on the circumference of a circle of which C is the centre; draw tangents at A and B meeting at T; and from A draw AN perpendicular to CB: shew that BT is to BC as BN is to NA.

**275.** In the sides AB, AC of a triangle ABC are taken two points D, E, such that BD is equal to CE; DE, BC are produced to meet at F: shew that AB is to AC as EF is to DF.

**276.** Find a point the perpendiculars from which on the sides of a given triangle shall be in a given ratio.

**277.** On AB, AC, two adjacent sides of a rectangle, two similar triangles are constructed, and perpendiculars are drawn to AB, AC from the angles which they subtend, intersecting at the point P. If AB, AC be homologous sides, shew that P is in all cases in one of the diagonals of the rectangle.

**278.** APB and CQD are parallel straight lines directed towards the same parts; and AP is to PB as CQ is to QD: shew that the straight lines PQ, AC, BD when produced will meet at a point.

**279.** ACB is a triangle, and the side AC is produced to D so that CD is equal to AC, and BD is joined: if any straight line drawn parallel to AB cuts the sides AC, CB, and from the points of section

straight lines be drawn parallel to DB, shew that these straight lines will meet AB at points equidistant from its extremities.

**280.** If a circle be described touching externally two given circles, the straight line passing through the points of contact will intersect the straight line passing through the centres of the given circles at a fixed point.

**281.** D is the middle point of the side BC of a triangle ABC, and P is any point in AD; through P the straight lines BPE, CPF are drawn meeting the other sides at E, F: shew that EF is parallel to BC.

**282.** AB is the diameter of a circle, E the middle point of the radius OB; on AE, EB as diameters circles are described; PQL is a common tangent meeting the circles at P and Q, and AB produced at L: shew that BL is equal to the radius of the smaller circle.

**283.** ABCD is a parallelogram; P and Q are points in a straight line parallel to AB; PA and QB meet at R, and PD and QC meet at S; shew that RS is parallel to AD.

**284.** A and B are two given points; AC and BD are perpendicular to a given straight line CD; AD and BC intersect at E, and EF is perpendicular to CD: shew that AF and BF make equal angles with CD.

**285.** From the angular points of a parallelogram ABCD perpendiculars are drawn on the diagonals meeting them at E, F, G, H respectively: shew that EFGH is a parallelogram similar to ABCD.

**286.** If at a given point two circles intersect, and their centres lie on two fixed straight lines which pass through that point, shew that whatever be the magnitude of the circles their common tangents will always meet in one of two fixed straight lines which pass through the given point.

**287.** A circle is described about an equilateral triangle ABC, and from A a straight line is drawn cutting BC and meeting the circle again in D; prove that BD, DC together are equal to AD.

## VI. 7-18.

**288.** Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in a given ratio.

**289.** In a given triangle draw a straight line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.

**290.** O is a fixed point in a given straight line OA, and a circle of given radius moves so as always to be touched by OA; a tangent OP

is drawn from  $O$  to the circle, and in  $OP$  produced  $PQ$  is taken a third proportional to  $OP$  and the radius: shew that as the circle moves along  $OA$ , the point  $Q$  will move in a straight line.

**291.** Find a point in a side of a triangle, from which two straight lines drawn, one to the opposite angle, and the other parallel to the base, shall cut off towards the vertex and towards the base, equal triangles.

**292.**  $ACB$  is a triangle having a right angle at  $C$ ; from  $A$  a straight line is drawn at right angles to  $AB$ , cutting  $BC$  produced at  $E$ ; from  $B$  a straight line is drawn at right angles to  $AB$ , cutting  $AC$  produced at  $D$ : shew that the triangle  $ECD$  is equal to the triangle  $ACB$ .

**293.** The straight line bisecting the angle  $ABC$  of the triangle  $ABC$  meets the straight lines drawn through  $A$  and  $C$ , parallel to  $BC$  and  $AB$  respectively, at  $E$  and  $F$ : shew that the triangles  $CBE$ ,  $ABF$  are equal.

**294.**  $AB$ ,  $CD$  are any two chords of a circle passing through a point  $O$ ;  $EF$  is any chord parallel to  $OB$ ; join  $CE$ ,  $DF$  meeting  $AB$  at the points  $G$  and  $H$ , and  $DE$ ,  $CF$  meeting  $AB$  at the points  $K$  and  $L$ : shew that the rectangle  $OG$ ,  $OH$  is equal to the rectangle  $OK$ ,  $OL$ .

**295.**  $ABCD$  is a quadrilateral in a circle; the straight lines  $CE$ ,  $DE$  which bisect the angles  $ACB$ ,  $ADB$  cut  $BD$  and  $AC$  at  $F$  and  $G$  respectively: shew that  $EF$  is to  $EG$  as  $ED$  is to  $EC$ .

**296.** From an angle of a triangle two straight lines are drawn, one to any point in the side opposite to the angle, and the other to the circumference of the circumscribing circle, so as to cut from it a segment containing an angle equal to the angle contained by the first drawn line and the side which it meets: shew that the rectangle contained by the sides of the triangle is equal to the rectangle contained by the straight lines thus drawn.

**297.** The vertical angle  $C$  of a triangle is bisected by a straight line which meets the base at  $D$ , and is produced to a point  $E$ , such that the rectangle contained by  $CD$  and  $CE$  is equal to the rectangle contained by  $AC$  and  $CB$ : shew that if the base and vertical angle be given, the position of  $E$  is invariable.

**298.** A square is inscribed in a right-angled triangle  $ABC$ , one side  $DE$  of the square coinciding with the hypotenuse  $AB$  of the triangle: shew that the area of the square is equal to the rectangle  $AD$ ,  $BE$ .

**299.**  $ABCD$  is a parallelogram; from  $B$  a straight line is drawn cutting the diagonal  $AC$  at  $F$ , the side  $DC$  at  $G$ , and the side  $AD$  produced at  $E$ : shew that the rectangle  $EF$ ,  $FG$  is equal to the square on  $BF$ .

**300.** If a straight line drawn from the vertex of an isosceles triangle to the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained by the whole line so produced, and the part of it between the vertex and the base shall be equal to the square on either of the equal sides of the triangle.

**301.** A triangle ABC is inscribed in a circle; a line BD drawn parallel to the tangent to the circle at A meets AC in D. Prove that the rectangles AB, BC and AC, BD are equal.

**302.** A square DEFG is inscribed in the right-angled triangle ABC so that the corners D, E lie on the hypotenuse AB and the other two corners on the sides BC, CA; prove that the square = the rect. AD, EB.

## VI. 19-D.

**303.** An isosceles triangle is described having each of the angles at the base double of the third angle: if the angles at the base be bisected, and the points where the lines bisecting them meet the opposite sides be joined, the triangle will be divided into two parts in the proportion of the base to the side of the triangle.

**304.** Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

**305.** Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

**306.** If two isosceles triangles are to one another in the duplicate ratio of their bases, shew that the triangles are similar.

**307.** From a point without a circle draw a straight line cutting the circle, so that the two segments shall be equal to each other.

**308.** In the figure of II. 11 shew that four other straight lines, besides the given straight line are divided in the required manner.

**309.** ABC is an isosceles triangle, the side AB being equal to AC; F is the middle point of BC; on any straight line through A perpendiculars FG and CE are drawn: shew that the rectangle AC, EF is equal to the sum of the rectangles FC, EG and FA, FG.

**310.** A, B, C, and D are four points in a straight line and O is any point at which AB and CD subtend equal angles; prove that  

$$BD \cdot CD : BA \cdot CA :: OD^2 : OA^2.$$

**311.** ABCD is a quadrilateral and AC, BD are the diagonals; prove that the centroids of the triangles ABD, CBD, ADC, ABC form a quadrilateral similar to ABCD and of one-ninth the area.

## VI. 1-D.

**312.** AB is a diameter, and P any point in the circumference of a circle; AP and BP are joined and produced if necessary; from any point C in AB a straight line is drawn at right angles to AB meeting AP at D and BP at E, and the circumference of the circle at F: shew that CD is a third proportional to CE and CF.

**313.** A, B, C are three points in a straight line, and D a point at which AB and BC subtend equal angles: shew that the locus of D is the circumference of a circle.

**314.** If a straight line be drawn from one corner of a square cutting off one-fourth from the diagonal it will cut off one-third from a side. Also if straight lines be drawn similarly from the other corners so as to form a square, this square will be two-fifths of the original square.

**315.** The sides AB, AC of a given triangle ABC are produced to any points D, E, so that DE is parallel to BC. The straight line DE is divided at F so that DF is to FE as BD is to CE: shew that the locus of F is a straight line.

**316.** A, B are two fixed points on the circumference of a given circle, and P is a moveable point on the circumference; on PB is taken a point D such that PD is to PA in a constant ratio, and on PA is taken a point E such that PE is to PB in the same ratio: shew that DE always touches a fixed circle.

**317.** ABC is an isosceles triangle, the angle at A being four times either of the others: shew that if BC be trisected at D and E, the triangle ADE is equilateral.

**318.** Perpendiculars are let fall from two opposite angles of a rectangle on a diagonal: shew that they will divide the diagonal into equal parts, if the square on one side of the rectangle be double that on the other.

**319.** A straight line AB is divided into any two parts at C, and on the whole straight line and on the two parts of it equilateral triangles ADB, ACE, BCF are described, the two latter being on the same side of the straight line, and the former on the opposite side; G, H, K are the centres of the circles inscribed in these triangles: shew that the angles AGH, BGK are respectively equal to the angles ADC, BDC, and that GH is equal to GK.

**320.** On the two sides of a right-angled triangle squares are described: shew that the straight lines joining the acute angles of the triangle and the opposite angles of the squares cut off equal segments from the sides, and that each of these equal segments is a mean proportional between the remaining segments.

**321.** Two straight lines and a point between them are given in position: draw two straight lines from the given point to terminate in the given straight lines, so that they shall contain a given angle and have a given ratio.

**322.** With a point A in the circumference of a circle ABC as centre, a circle PBC is described cutting the former circle at the points B and C; any chord AD of the former meets the common chord BC at E, and the circumference of the other circle at O: shew that the angles EPO and DPO are equal for all positions of P.

**323.** ABC, ABF are triangles on the same base in the ratio of two to one; AF and BF produced meet the sides at D and E; in FB a part FG is cut off equal to FE, and BG is bisected at O: shew that BO is to BE as DF is to DA.

**324.** A is the centre of a circle, and another circle passes through A and cuts the former at B and C; AD is a chord of the latter circle meeting BC at E, and from D are drawn DF and DG tangents to the former circle: shew that G, E, F lie on one straight line.

**325.** In AB, AC, two sides of a triangle, are taken points D, E; AB, AC are produced to F, G such that BF is equal to AD, and CG equal to AE; BG, CF are joined meeting at H: shew that the triangle FHG is equal to the triangles BHC, ADE together.

**326.** In any triangle ABC if BD be taken equal to one-fourth of BC, and CE one-fourth of AC, the straight line drawn from C through the intersection of BE and AD will divide AB into two parts, which are in the ratio of nine to one.

**327.** Any rectilineal figure is inscribed in a circle: shew that by bisecting the arcs and drawing tangents to the points of bisection parallel to the sides of the rectilineal figure, we can form a rectilineal figure circumscribing the circle equiangular to the former.

**328.** Find a mean proportional between two similar right-angled triangles which have one of the sides containing the right angle common.

**329.** In the sides AC, BC of a triangle ABC points D and E are taken, such that CD and CE are respectively the third parts of AC and BC; BD and AE are drawn intersecting at O; shew that EO and DO are respectively the fourth parts of AE and BD.

**330.** CA, CB are diameters of two circles which touch each other externally at C; a chord AD of the former circle, when produced, touches the latter at E, while a chord BF of the latter, when produced, touches the former at G; shew that the rectangle contained by AD and BF is four times that contained by DE and FG.



**331.** Two circles intersect at A, and BAC is drawn meeting them at B and C; with B, C as centres are described two circles each of which intersects one of the former at right angles; shew that these circles and the circle whose diameter is BC meet at a point

**332.** ABCDEF is a regular hexagon; shew that BF divides AD in the ratio of one to three, and that EC is trisected by FD, BD.

**333.** ABC, DEF are triangles, having the angle A equal to the angle D, and AB is equal to DF; shew that the areas of the triangles are as AC to DE.

**334.** If M, N be the points at which the inscribed and an escribed circle touch the side AC of a triangle ABC; show that if BM be produced to cut the escribed circle again at P, then NP is a diameter.

**335.** The angle A of a triangle ABC is a right angle, and D is the foot of the perpendicular from A on BC; DM, DN are perpendiculars on AB, AC; shew that the angles BMC, BNC are equal.

**336.** If from the point of bisection of any given arc of a circle two straight lines be drawn, cutting the chord of the arc and the circumference, the four points of intersection shall also lie in the circumference of a circle.

**337.** The side AB of a triangle ABC is touched by the inscribed circle at D, and by the escribed circle at E: shew that the rectangle contained by the radii is equal to the rectangle AD, DB and also to the rectangle AE, EB.

**338.** A parallelogram is inscribed in a triangle, having one side on the base of the triangle, and the adjacent sides parallel to a fixed direction: shew that the locus of the intersection of the diagonals of the parallelogram is a straight line bisecting the base of the triangle.

**339.** From a given point outside two given circles which do not meet, draw a straight line such that the portions of it intercepted by each circle shall be respectively proportional to their radii.

**340.** In a given triangle inscribe a rhombus which shall have one of its angular points coincident with a point in the base, and a side on that base.

**341.** ABC is a triangle having a right angle at C; ABDE is the square described on the hypotenuse; F, G, H are the points of intersection of the diagonals of the squares on the hypotenuse and sides; shew that the angles DCE, GFH are together equal to a right angle.

## XI. 1-21.

**342.**  $ABC$  is a triangle; the perpendiculars from  $A$  and  $B$  on the opposite sides meet at  $D$ ; through  $D$  a straight line is drawn perpendicular to the plane of the triangle, and  $E$  is any point in this straight line; shew that the straight line joining  $E$  to any angular point of the triangle is at right angles to the straight line drawn through that angular point parallel to the opposite side of the triangle.

**343.** From a point  $E$  draw  $EC$ ,  $ED$  perpendicular to two planes  $CAB$ ,  $DAB$  which intersect in  $AB$ , and from  $D$  draw  $DF$  perpendicular to the plane  $CAB$  meeting it at  $F$ : show that the straight line  $CF$ , produced if necessary, is perpendicular to  $AB$ .

**344.**  $BCD$  is the common base of two pyramids, whose vertices  $A$  and  $E$  lie in a plane passing through  $BC$ ; and  $AB$ ,  $AC$  are respectively perpendicular to the faces  $BED$ ,  $CED$ ; shew that one of the angles at  $A$  together with the angles at  $E$  make up four right angles, supposing  $A$  and  $E$  to be on opposite sides of  $BC$ .

**345.** Within the area of a given triangle is inscribed another triangle: shew that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles is less than the sum of the angles subtended at the same point by the sides of the exterior angle.

**346.** Shew that the perpendicular drawn from the vertex of a regular tetrahedron on the opposite face is three times that drawn from its own foot on any of the other faces.

**347.** A triangular pyramid stands on an equilateral base and the angles at the vertex are right angles: shew that the sum of the perpendiculars on the faces from any point of the base is constant.

**348.** Three straight lines not in the same plane intersect at a point, and through their point of intersection another straight line is drawn within the solid angle formed by them; shew that the angles which this straight line makes with the first three are together less than the sum, but greater than half the sum, of the angles which the first three make with each other.

**349.** Three straight lines which do not all lie in one plane, are cut in the same ratio by three planes, two of which are parallel: shew that the third will be parallel to the other two, if its intersections with the three straight lines are not all in the same straight line.

**350.** The straight line  $PBbp$  cuts two parallel planes at  $B$ ,  $b$ , and the points  $P$ ,  $p$  are equidistant from the planes;  $PAa$ ,  $pcC$  are other straight lines drawn from  $P$ ,  $p$  to cut the planes: shew that the triangles  $ABC$ ,  $abc$  are equal.



**351.** The locus of the points which are equidistant from two given points is a plane.

**352.** The locus of the points which are equidistant from three given points is a straight line which is perpendicular to the plane through the three points.

**353.** The locus of the straight lines which cut a given straight line perpendicularly at a given point in it is a plane.

**354.** The locus of the points which are equidistant from two given intersecting straight lines is a pair of planes.

**355.** The locus of the points which are equidistant from three straight lines, which meet in a point but are not in the same plane, is four straight lines.

**356.** P is any point in a plane and O a point outside the plane. On OP is taken a point Q such that the rectangle  $OP \cdot OQ$  is constant; the locus of Q is a sphere passing through O.

**357.** Through a given point in space draw a straight line to meet each of two given straight lines which are not in the same plane.

**358.** From a point A two perpendiculars AP, AQ are drawn to each of two intersecting planes; prove that the common section of these planes is perpendicular to the plane through AP, AQ.

**359.** PA, PB, PC are three concurrent and mutually perpendicular straight lines, and PO is perpendicular to the plane ABC; shew that O is the orthocentre of the triangle ABC.

**360.** OA, OB, OC are three mutually perpendicular straight lines; from O perpendiculars OP, OQ, OR are let fall on BC, CA, AB. Prove that BC bisects the angle QPR externally.

**361.** The faces of a parallelopiped are parallelograms, and those which are opposite are identically equal.

**362.** The four diagonals of a parallelopiped have a common point and bisect one another.

**363.** Any section of a parallelopiped made by a plane cutting two opposite pairs of faces and not the remaining pair is a parallelogram.

**364.** The sum of the squares on the diagonals of a parallelopiped equals the sum of the squares on its edges.

**365.** Prove that four planes can be drawn through the centre of a cube, each of which cuts six edges of the cube in the corners of a regular hexagon and the other six sides produced in the corners of another regular hexagon.

**366.** If the sides of a skew quadrilateral ABCD (that is, a quadrilateral no three sides of which are in the same plane) be cut by a plane in the four points  $a, b, c, d$ , prove that

$$aA \cdot bB \cdot cC \cdot dD = aB \cdot bC \cdot cD \cdot dA.$$

**367.** The areas of the sections of a pyramid made by planes parallel to the base are in the duplicate ratio of the perpendicular distances of the sections from the vertex.

**368.** If the three plane angles at the vertex of a tetrahedron be bisected and the points in which the bisecting lines meet the sides of the base be joined with the opposite angles of the base, the three lines so formed will meet in a point.

**369.** The sum of the squares on the edges of a tetrahedron are equal to four times the squares on the straight lines joining the middle points of opposite edges.

**370.** The sum of the squares on any two opposite edges of a tetrahedron is less than the sum of the squares on the other four edges.

**371.** The shortest distance between two opposite edges of a regular tetrahedron equals half the diagonal of the square on an edge.

**372.** In any tetrahedron the straight lines joining each vertex to the centroid of the opposite face meet in a point and divide one another in the ratio 3 : 1.

**373.** If ABCD be a regular tetrahedron, and O its centre, prove that the radii of the spheres ABCD and OBCD are as 2 to 3.

**374.** A tetrahedron is cut by a plane so that the section is a rhombus; prove that the side of the rhombus is half the harmonic mean between a pair of opposite edges.

**375.** If the areas of all the faces of a tetrahedron are equal each edge is equal to the opposite edge.

**376.** If an equifacial tetrahedron be cut by a plane parallel to two edges which do not meet, the perimeter of the parallelogram so formed is double of either edge of the tetrahedron to which it is parallel.

**377.** If through the middle point of each edge of a tetrahedron a plane is drawn perpendicular to the opposite edge, prove that the six planes thus obtained meet in a point.

**378.** The six planes through the middle points of the edges of a tetrahedron and perpendicular to the edges meet in a point.

**379.** The straight lines which join the middle points of the opposite edges of a tetrahedron meet in a point.

**380.** In any tetrahedron the line joining the middle points of one pair of opposite edges is perpendicular to the shortest line between either of the other pair of opposite edges.

**381.** If the sum of the squares on opposite edges of a tetrahedron is the same for each pair, then the perpendiculars from the angular points on the opposite faces meet in a point.

**382.** If the perpendiculars from two of the angular points of a tetrahedron on the opposite faces meet in a point, the perpendiculars from the other two angular points meet in a point.

**383.** If the perpendiculars to two faces of a tetrahedron drawn from the opposite vertices meet, the edge in which the faces intersect is perpendicular to the opposite edge.

**384.** If the line joining any vertex to the orthocentre of the opposite face be perpendicular to that face, the same is true with regard to the other vertices.

If two pairs of opposite edges of a tetrahedron be at right angles to each other prove that

**385.** the other pair is at right angles also.

**386.** their six middle points lie on a sphere.

**387.** the sum of the squares on each pair of opposite edges is the same.

**388.** the four perpendiculars from the vertices upon the opposite faces, and the three shortest distances between opposite edges are concurrent.

**389.** If the circles inscribed in three of the faces of a tetrahedron touch one another, the circle inscribed in the fourth face will touch the other three.

## MISCELLANEOUS.

**390.** Find the condition that must exist so that it may be possible to fold down the four corners of a quadrilateral flat down on the paper so that the four angular points may meet in a point and the paper be everywhere doubled.

**391.** The straight lines joining the middle points of opposite sides of a quadrilateral, and the straight line joining the middle points of its diagonals meet in a point.

**392.** The area of the greatest parallelogram that can be inscribed in a given  $\triangle$  is one half of the area of the  $\triangle$ .

**393.** Prove that I. 47 can be proved by cutting off four right-angled  $\triangle$ 's from each of two equal squares.

**394.** ABC is a given equilateral triangle, and in the sides BC, CA, AB the points X, Y, Z are taken so that BX, CY, AZ are equal. AX, BY, and CZ are drawn so as to form a triangle; prove that it is equilateral.

**395.** ABCD is a parallelogram, and on AB, BC, CD, DA are taken points P, Q, R, S, such that AP, BQ, CR, DS are all equal; shew that PQRS is a parallelogram.

**396.** The centre C of a circle BPQ lies on another circle APQ, of which PBA is a diameter. Prove that PC and BQ are parallel.

**397.** The diagonals of a quadrilateral inscribed in a circle are at right angles; prove that the sum of the squares on either pair of opposite sides equals the square on the diameter.

**398.** A quadrilateral is inscribed in a circle; from its centre perpendiculars are drawn to the sides and a second quad<sup>l</sup> formed by joining their feet. Prove that the area of the second quad<sup>l</sup> is half that of the first.

**399.** Three circles touch one another externally at A, B, C; the chords AB, AC of two of them are produced to meet the third again in the points D, E; prove that DE is the diameter of the third circle and is parallel to the line joining the centres of the others.

**400.** Through P one of the points of intersection of two circles APB, APC a straight line BC is drawn perp<sup>r</sup> to AP and BA, CA meet the circles in Q and R; prove that AP bisects the  $\angle QPR$ .

**401.** If two circles pass through the vertex and a point in the bisector of an angle they intercept equal lengths on the sides.

**402.** A'BC, B'CA, and C'AB are equilateral  $\Delta^s$  described on the sides of a  $\Delta$  ABC with their vertices outside the  $\Delta$ . Prove that AA', BB', CC' are equal and meet in a point.

**403.** A, B, C are three points on a circle whose centre is O such that the arc AB is twice the arc BC. If AH, BK be drawn  $\perp^r$  to OB, OC prove that HK and AB are parallel.

**404.** The arc BC of a circle ABC, of centre O, is double of the arc AB. If P be the middle point of the arc BC, and CP, AB be produced to meet at E, prove that a circle can be described about the quadrilateral BECO.

**405.** Two circles touch at A; through A are drawn two lines BAC B'AC' to meet the circles in B, C and B', C'; prove that BB', and CC', are parallel, and also that the tangents at B and C are parallel and also those at B', C'.

**406.** Through one of the points of intersection of two circles draw a straight line that shall be bisected in that point and terminated by the circumference.

**407.** If a quad<sup>l</sup> be inscribed in a circle and the middle points of the arcs subtended by its sides be joined to make another quadrilateral and so on; prove that these quadrilaterals tend to become squares.

**408.** Two circles meet in  $P$  and  $Q$ . A straight line  $MPN$  is drawn terminated by the circles in  $M$  and  $N$ . At  $M$  and  $N$  tangents are drawn to the circle which intersect in  $T$ . Prove that  $M$ ,  $N$ ,  $Q$ , and  $T$  all lie on a circle.

**409.** If the tangent  $PT$  at any point  $P$  of a circle meets a diameter  $AB$  produced at  $T$  and if  $P$  be joined to  $B$ , the extremity of the diameter nearer to  $T$ , prove that the  $\angle ATP$  is equal to the complement of twice the  $\angle BPT$ .

**410.** The sides  $AB$ ,  $AC$  of a  $\triangle$  are bisected in  $E$ ,  $F$ .  $EF$  cuts the circle on  $AB$  as diameter in  $H$  and  $K$ . Prove that  $BH$  and  $BK$  are the internal and external bisectors of the angle  $ABC$ .

**411.**  $ABCD$  are four collinear points and  $EF$  is a common tangent to the circles described on the lines  $AB$ ,  $CD$  as diameters; prove that the triangles  $AEB$ ,  $CFD$  are equiangular.

**412.** Three circles touch the sides of a  $\triangle ABC$  in the points  $D$ ,  $E$ ,  $F$ , in which the in-circle touches the sides and touch one another in  $G$ ,  $H$ ,  $K$ . Prove that  $AG$ ,  $BH$ ,  $CK$  meet in a point.

**413.** The difference in the squares on the direct and transverse common tangents to two circles is equal to the product of their diameters.

**414.** The centre  $A$  of a circle lies on another circle which cuts the former in  $B$  and  $C$ ;  $AD$  is a chord of the latter circle cutting  $BC$  in  $E$ ; from  $D$  are drawn tangents  $DF$ ,  $DG$  to the former circle; prove that  $F$ ,  $E$ ,  $G$  are in one straight line.

**415.** If the sides of a cyclic quadrilateral touch a circle the lines joining the opposite points of contact are at right angles.

**416.** The circles which have for chords the sides of a cyclic quadrilateral intersect again in four concyclic points.

**417.** The sides  $AB$ ,  $AC$  of a triangle are produced and the exterior angles are bisected by straight lines which meet in  $O$ ; prove that the centre of the circumcircle of the triangle  $BOC$  lies on the circumcircle of  $ABC$ .

**418.** A straight line  $AB$  is divided into three equal parts at  $C$  and  $D$  and  $CPD$  is an equilateral triangle. Prove that  $AP$  is a tangent to the circle  $BPC$ .

**419.**  $AD$  bisects the angle  $A$  of a triangle and meets the base  $BC$  in  $D$  and a circle is described passing through  $A$  and touching  $BC$  at  $D$ . Prove that this circle will touch at  $A$  the circumcircle of  $ABC$ .

**420.**  $ABC$  is an equilateral triangle and  $A'$ ,  $B'$ ,  $C'$  points on its sides such that  $BA'$ ,  $CB'$ ,  $AC'$  are equal; prove that the circles about  $ABC$ ,  $A'B'C'$  are concentric and that the circles about the triangles  $AB'C'$ ,  $BC'A'$ ,  $CA'B'$  intersect in the common centre of the first two.

**421.** The straight line joining the orthocentre of a triangle to the middle point of the base meets the circumcircle in  $K$ , and the perpendicular from the vertex upon the base meets the circumcircle in  $G$ ; prove that  $KG$  is parallel to the base.

**422.**  $O$  being the orthocentre of a triangle  $ABC$ , prove that the circumcircles of the triangles  $BOC$ ,  $COA$ ,  $AOB$  are equal to that of  $ABC$ .

**423.** Three circles whose centres are  $A$ ,  $B$ , and  $C$  touch one another externally at  $D$ ,  $E$ , and  $F$ ; prove that the in-circle of the triangle  $ABC$  is the circumcircle of the triangle  $DEF$ .

**424.**  $ABC$  is an isosceles triangle and on the base  $BC$ , or  $BC$  produced, a point  $D$  is taken; prove that the circumcircles of the triangles  $ABD$ ,  $ACD$  are equal.

**425.** A circle passing through  $B$ ,  $C$  and the in-centre of the  $\triangle ABC$  meets the sides  $AB$ ,  $AC$  again in  $E$  and  $F$ ; prove that  $EF$  touches the in-circle.

**426.** The in-circle of a triangle  $ABC$  touches  $AC$ ,  $AB$  in  $E$  and  $F$ . Prove that the circle on  $BC$  as diameter meets the bisectors of the  $\angle$ s  $B$ ,  $C$  in points lying on the straight line  $EF$ .

**427.** The point in which the external bisector of one angle of a triangle again cuts the circumcircle is equidistant from the other two angular points of the triangle and from the centres of two escribed circles.

**428.** If the bisector of the vertical angle  $A$  of a triangle  $ABC$  meets the circumcircle of the triangle in  $D$  and  $DE$ ,  $DF$  be drawn  $\perp$  to the sides  $AB$ ,  $AC$ , produced if necessary, prove that  $AE$  equals the semi-sum of the sides.

**429.** Compare the areas of a square and an equilateral  $\triangle$  inscribed in the same circle.

**430.** In a regular heptagon the ends,  $D$  and  $E$ , of a side are joined to the opposite vertex  $A$ ; prove that in the  $\triangle ADE$  each angle at the base is three times the vertical angle.

**431.** If  $I$  be the in-centre of a  $\triangle ABC$  and  $D$ ,  $E$ ,  $F$  the circumcentres of the  $\triangle$ s  $BIC$ ,  $CIA$ ,  $AIB$  respectively then the  $\triangle DEF$  is equiangular to  $I_1I_2I_3$  and has the same circumcircle as  $ABC$ .

**432.**  $ABCD$  is a quadrilateral inscribed in a circle and  $BD$  bisects  $AC$ ; prove that the rectangles  $AD$ ,  $AB$  and  $DC$ ,  $CB$  are equal.

**433.**  $AC$  and  $BD$  are the diagonals of a quadrilateral  $ABCD$ ; prove that the sum of the perpendiculars from  $A$  and  $C$  on  $BD$  is to the sum of the perpendiculars from  $B$  and  $D$  on  $AC$  as  $AC$  to  $BD$ .

**434.**  $AB$  is any fixed straight line and  $CD$  a chord of a circle parallel to  $AB$ ;  $AC$  being joined cuts the circle in  $E$  and  $BE$  meets it

in F. Prove that DF will cut AB in a point G which is the same for all chords.

**435.** PHQ and POQ are two triangles on opposite sides of the same base PQ such that the angles QHP and HPQ are equal to the angles OQP and QPO respectively. Any parallel to OQ cuts PO and PQ in L and M. Prove that QL and HM meet on a fixed circle.

**436.** If ABC be a triangle, right-angled at A, and AD be the  $\perp$  on BC, prove that

$$\frac{1}{AD^2} = \frac{1}{AB^2} + \frac{1}{AC^2}.$$

**437.** Two points, P and Q, are taken on two given straight lines AB, AC such that the sum of AP, AQ is constant. Prove that the circle PAQ goes through a second fixed point.

**438.** From the vertex of a right-angled triangle a perpendicular is drawn on the hypotenuse, and from the foot of this perpendicular another is drawn on each side of the triangle; shew that the area of the triangle of which these two latter perpendiculars are two of the sides cannot be greater than one-fourth of the area of the original triangle.

**439.** If the extremities of two intersecting straight lines be joined so as to form two vertically opposite triangles, the figure made by connecting the points of bisection of the given straight lines, will be a parallelogram equal in area to half the difference of the triangles.

**440.** AB, AC are two tangents to a circle, touching it at B and C; R is any point in the straight line which joins the middle points of AB and AC; shew that AR is equal to the tangent drawn from R to the circle.

**441.** AB, AC are two tangents to a circle; PQ is a chord of the circle which, produced if necessary, meets the straight line joining the middle points of AB, AC at R; shew that the angles RAP, AQR are equal to one another.

**442.** Shew that the four circles each of which passes through the middle points of the sides of one of the four triangles formed by two adjacent sides and a diagonal of any quadrilateral all intersect at a point.

**443.** Perpendiculars are drawn from any point on the three straight lines which bisect the angles of an equilateral triangle: shew that one of them is equal to the sum of the other two.

**444.** Two circles intersect at A and B, and CBD is drawn through B perpendicular to AB to meet the circles; through A a straight line is drawn bisecting either the interior or exterior angle between AC and AD, and meeting the circumferences at E and F: shew that the tangents to the circumferences at E and F will intersect in AB produced.



**445.** Divide a triangle by two straight lines into three parts, which, when properly arranged, shall form a parallelogram whose angles are of given magnitude.

**446.** ABCD is a parallelogram, and P is any point: shew that the triangle PAC is equal to the difference of the triangles PAB and PAD, if P is within the angle BAD or that which is vertically opposite to it; and that the triangle PAC is equal to the sum of the triangles PAB and PAD, if P has any other position.

**447.** Two circles cut each other, and a straight line ABCDE is drawn, which meets one circle at A and D, the other at B and E, and their common chord at C; shew that the square on BD is to the square on AE as the rectangle BC, CD is to the rectangle AC, CE.

**448.** AD, BE, CF are the perpendiculars of a triangle ABC. If from D, E, F are drawn perpendiculars to the adjacent sides the feet of these six perpendiculars all lie on a circle.

[This circle is called TAYLOR'S CIRCLE.]

**449.** D is a point on the side BC of a triangle ABC, and EDF is a transversal through it; prove that the locus of the second point of intersection of the circumcircles of the triangles BDE, FDC is a circle.

**450.** If through any point on the radical axis of two circles a diameter be drawn to each circle, prove that the centre of the circle through the extremities of these diameters lies on a fixed line.

**451.** Circles are drawn with centres on a fixed line to cut a given circle orthogonally; prove that they all pass through two given points.

**452.** Any point E is taken outside a given circle and F is a point on its polar; prove that the circles whose centres are E, F and which cut the given circle orthogonally cut one another orthogonally.

**453.** If A', B', C' be the midpoints of the sides of a triangle ABC, prove that the in-centre of A'B'C' is collinear with the in-centre and centroid of the triangle ABC.

**454.** If from a point without a circle any straight lines APQ, ARS be drawn, making equal angles with the diameter through A and cutting the circle in P, Q, R, S respectively, prove that PS and QR meet in a fixed point.

**455.** If C be any point within a circle, centre O, and if CD, CE be drawn on the same side of OC, making equal angles with OC and cutting the circle in D and E, prove that the chord DE passes through a fixed point.

**456.** If G be the centroid of a triangle ABC, then tangents from A, B, C respectively to the circumcircles of GBC, GCA, GAB are all equal.



**457.** The locus of a point such that the rectangle contained by its distances from an opposite pair of sides of a cyclic quadrilateral is equal to the rectangle contained by its distances from the other pair is the circumcircle of the quadrilateral.

**458.** A and B are two fixed points on a fixed circle whose centre is C and QCR is any diameter; the circles described round ACQ and BCR will intersect on a circle cutting the fixed circle at right angles in A and B.

**459.** PAQ is a chord drawn through A, one of the points of intersection of two given circles, to meet them in P, Q; the locus of the middle point of PQ is a circle.

**460.** The locus of the foot of the perp<sup>r</sup> from a fixed point upon a chord of a circle which subtends a right  $\angle$  at it is another circle.

**461.** Straight lines are drawn through the middle points of the sides of a  $\triangle$  perpendicular to the bisectors of the opposite angle. Prove that the triangle formed by these straight lines has the same nine-point circle as the original  $\triangle$ .

**462.** O is the orthocentre of a  $\triangle ABC$  and O' is its circumcentre; A', B', C' are the centres of the circumcentres of the  $\triangle^s$ . BOC, COA, AOB respectively. Prove that O' is the orthocentre of A'B'C', O its circumcentre, A, B, C the centres of the circumcircles of B'O'C', C'O'A' and A'O'B' respectively and that all the eight  $\triangle^s$  mentioned above have the same nine-point circle.

**463.** AE, AD are the median line and perpendicular respectively from A to the base BC of a  $\triangle ABC$  and P, H, K are the orthocentres of the  $\triangle^s$  ABC, AEC, ABE; prove that

(1) PHDK is a harmonic range;

(2) the nine-point circle of ABC and the circle EHK cut orthogonally.

**464.** Of the four triangles formed by the centres of the four circles touching the sides of any triangle, the centroid of any one is the orthocentre of the centroids of the other three.

**465.** P is a point on the circumcircle of the triangle ABC and PL, PM are the perpendiculars on the sides BC, CA and PY the perpendicular on the pedal line LM; prove that  $PY \cdot PC = PL \cdot PM$ .

**466.** The poles of the radical axis of two circles with respect to the circles are the harmonic conjugates with respect to the centres of similitude.

**467.** In a triangle if any two of the four points viz. the orthocentre, the centroid, the in-centre and the circumcentre coincide the triangle is equilateral.





